Nonlinear Elasticity:
An Existence theorem for a simple Elastic model

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1 Introduction

In these notes we prove an existence theorem for a 1d model of hyperelasticity. A crucial property that helps proving such existence theorems is the *lower semicontinuity* of the total energy functional. As we shall show, a sufficient condition for the total energy to be lower semicontinuous is convexity of the stored energy density. For the various reasons discussed in lecture (or refer to [1]), convexity of the stored energy functional is not appropriate for 3d-elasticity. Ball, in his seminal paper [2] observed that polyconvexity of the stored energy functional implies lower semicontinuity of the total energy. In 1d, however, polyconvexity and convexity are equivalent and consequently the proof of existence is significantly simpler.

The existence theorem presented in this report relies on the *direct method* in the calculus of variations [3, 4]. In order to rigorously understand this method we will need to take a detour into certain concepts of functional analysis [5].

These notes are mainly based on [1] and [3]. For more details about the functional analysis results, refer to [5].

2 Banach spaces and Weak topologies

Definition 1 [Banach space]**.** *A vector space along with a norm,* $(V, \|\cdot\|)$ *is a Banach space if it is complete with respect to the norm-induced metric.*

Let V be a Banach space over $\mathbb{R}$ and denote its dual,

$$V^* := \{L : V \to \mathbb{R} : L \text{ is continuous}\}.$$

The weak topology is the coarsest topology on V such that all the continuous linear functionals remain continuous. We define the criterion for sequence convergence in the weak topology:

Definition 2 [Weak convergence]**.** *A sequence $\{v_k\}$ is said to weakly converge to $v \in V$ iff $L(v_k) \to L(v)$ for all $L \in V^*$.*

We denote weak convergence in the following way:

$$v_k \rightharpoonup v.$$

For infinite dimensional spaces the weak topology is always distinct from the topology induced by the norm. In order to emphasise this difference, we will often use the term *strong topology* to refer to the latter.

It is straightforward to see that strong convergence $\implies$ weak convergence and consequently, weak closure $\implies$ strong closure. A sufficient condition for the converse to hold is the following

Theorem 1. *Let $C \subset V$ be a convex set. Then weak closure $\Leftrightarrow$ strong closure.*

The proof of the above result relies on the geometric Hahn-Banach theorem, so we will not be presenting it here.

The following result is crucial

Proposition 1. *In a reflexive Banach space (i.e. the evaluation map $V \to V^{**}$ is an isomorphism) a bounded sequence has a weakly convergent subsequence.*

3 Lower Semi-continuity

Definition 3 [Epigraph of a function]**.** *For a function $f : V \to \mathbb{R}$, the epigraph is defined to be*

$$\text{epi}(f) := \{(v, y) : f(v) \leq y\}$$

Intuitively, the epigraph is everything lying above the graph of a function. We can now define lower semi-continuity and sequential lower semi-continuity,

Definition 4. *A function, f is said to be lower semi-continuous if* $\mathrm{epi}(f)$ *is closed in $V \times \mathbb{R}$. A function is sequentially lower semi-continuous if*

$$f(v) \leq \liminf_{n\to\infty} f(v_n)$$

for every sequence v_n that converges to v.

Theorem 2 (Tonelli)**.** *Let h be a positive, convex, continuous function on $\mathbb{R}$ i.e.,*

$$h : (0, \infty) \to [0, \infty),$$

Let $\mathcal{V} := \{v \in L^1(0,1) : v > 0 \ \text{ a.e. in } \ (0,1)\}$, then the functional $H : \mathcal{V} \to \mathbb{R}$ defined by

$$H(F) = \int_0^1 h(F(X))\,\mathrm{d}X$$

is weakly lower semi-continuous.

Proof. First we will show that H is strongly lower semi-continuous on $\mathcal{V}$ and then augment the result to show weak lower semi-continuity.

Since the strong topology is metrizable, lower semi-continuity and sequential lower semi-continuity are equivalent. So let $F_k \to F$ in $\mathcal{V}$. We want to show that

$$\int_0^1 h(F(X))\,\mathrm{d}X \leq \liminf_{k\to\infty} \int_0^1 h(F_k(X))\,\mathrm{d}X.$$

First let us pass through a subsequence F_l to get

$$\lim_{l\to\infty} H(F_l) = \liminf_{k\to\infty} H(F_k).$$

Since $F_l \to F$ in L^1, there exists a subsequence $F_m \to F$ a.e. $X \in (0,1)$. Since h is continuous, we have $h(F_m(X)) \to h(F(X))$ a.e. $X \in (0,1)$. Using Fatou's lemma, then

$$\begin{aligned}
\int_0^1 h(F)\,\mathrm{d}X &= \int_0^1 \lim_{m\to\infty} h(F_m)\,\mathrm{d}X \\
&\leq \liminf_{m\to\infty} \int_0^1 h(F_m)\,\mathrm{d}X \\
&= \lim_{l\to\infty} \int_0^1 h(F_l)\,\mathrm{d}X \\
&= \liminf_{k\to\infty} \int_0^1 h(F_k)\,\mathrm{d}X.
\end{aligned}$$

It is also not difficult to see that H is convex on $\mathcal{V}$. Let $F_1, F_2 \in \mathcal{V}$

$$\begin{aligned} H(\lambda F_1 + (1-\lambda)F_2) &= \int_0^1 h(\lambda F_1 + (1-\lambda)F_2)\,\mathrm{d}X \\ &\leq \int_0^1 \lambda h(F_1) + (1-\lambda)h(F_2)\,\mathrm{d}X \\ &= \lambda H(F_1) + (1-\lambda)H(F_2) \end{aligned}$$

Since H is strongly lower semi-continuous and convex, epi(H) is strongly closed and convex, therefore it is weakly closed (from Theorem 1) and H is weakly lower semi-continuous. □

4 The Direct Method

Given an energy functional, $E : V \to \mathbb{R}$ that is bounded below, our objective is to minimise it over a set $\mathcal{A} \subset V$, which we will refer to as the 'admissible set'. Let $m := \inf_{w\in\mathcal{A}} E[w] > -\infty$ and choose $f_k \in \mathcal{A}$ such that $E[f_k] \to m$. We call such a sequence an 'infimising' sequence. Suppose f_k had a convergent subsequence with a limit in $\mathcal{A}$. Using this, along with some continuity assumption on E we could conclude the existence of a minimiser for E in $\mathcal{A}$.

We have two issues here: the first is that in order to get a convergent subsequence, we need some kind of compactness which is not easy in an infinite dimensional space. The second is that it is not clear what kind of continuity assumption is required on E. The answer in both cases lies in the weak topology.

Given certain growth conditions on the functional, it is possible to show that any infimising sequence is bounded. Provided V is reflexive, f_k has a subsequence that converges in the weak topology i.e. $f_{k_n} \rightharpoonup f_0$. In this case, it is sufficient for E to be sequentially weak lower semi-continuous for the existence of a minimiser:

$$E[f_0] \leq \liminf_{n\to\infty} E[f_{k_n}] \leq m,$$

where the last inequality follows from $E[f_k] \to m$. But m is the inf of E over $\mathcal{A}$, so $m \leq E[f_0]$, thus $E[f_0] = m$. If $\mathcal{A}$ is weakly closed, we may conclude that $f_0 \in \mathcal{A}$ is a minimiser of E.

5 Existence theorem

We apply the direct method to a 1d hyperelastic model to show the existence of a minimiser to the energy. Consider as the reference configuration the open interval $(0, 1)$. We will denote positions in the reference configuration by X. We look at maps that take the reference configuration to the interval (a, b) where $b > a$ with the following properties:

- $f(0) = a$ and $f(1) = b$,
- $\|f\|_{W^{1,p}(0,1)} < \infty$ or in other words, f is in the Sobolev space $W^{1,p}(0, 1)$. The norm above is defined as

  $$\|f\|_{W^{1,p}(0,1)} = \left(\int_0^1 |f|^p \, \mathrm{d}X + \int_0^1 |f'|^p \, \mathrm{d}X\right)^{1/p}$$

  with $p > 1$. The derivative f' is interpreted in the generalised sense,
- $F(X) \equiv f'(X) > 0$ a.e. $X \in (0, 1)$. This implies that f is globally invertible.

Our admissible set is therefore

$$\mathcal{A} := \{f \in W^{1,p}(0, 1) : f(0) = a, \ f(1) = b, \ F(X) > 0 \text{ a.e.}\}$$

We have the total energy functional,

$$E[f] = \int_0^1 W(F(X)) \, \mathrm{d}X - \int_0^1 b(X)f(X) \, \mathrm{d}X.$$

The first term here represents total stored energy and the second term is due to body forces (dead load). $b(X)$ is assumed to be continuous on $[0, 1]$. W is continuous, convex, positive and satisfies the growth condition $W(F) \to \infty$ as $F \to 0$ or ∞. In the latter case we specifically require that for $k > 0$ and $p > 1$:

$$W(F) \geq C + k\,|F|^p\,. \tag{$*$}$$

The above condition is called *coercivity.* Note that the p in the definition of the admissible set $\mathcal{A}$ is chosen *a posteriori* to equal the p in $(*)$. Also assume that $\inf_{f\in\mathcal{A}} E[f] < \infty$

Theorem 3. *Given the above assumptions, there exists $f_0 \in \mathcal{A}$ that minimises E.*

Proof. First we show that the loading term is a bounded linear functional on $W^{1,p}(0,1)$. Linearity is obvious. From the Morrey-Sobolev embedding theorem $W^{1,p}(0,1)$ is embedded in $C^0(0,1)$ so $\|f\|_{C^0} \leq C\|f\|_{W^{1,p}}$.

$$\int_0^1 bf \, \mathrm{d}X \leq \sup_{(0,1)} |f| \int_0^1 b(X) \, \mathrm{d}X = \tilde{C}_1 \|f\|_{C^0} \leq C_1 \|f\|_{W^{1,p}},$$

where $C_1 > 0$. Next we will use the coercivity inequality to show that E is bounded below,

$$\begin{aligned} E[f] &= \int_0^1 W(F) \, \mathrm{d}X - \int_0^1 bf \, \mathrm{d}X \\ &\geq \int_0^1 |F|^p + C \, \mathrm{d}X - C_1 \|f\|_{W^{1,p}} && \text{(from } (*)\text{)} \\ &\geq C_3 \|f\|_{W^{1,p}}^p - C_1 \|f\|_{W^{1,p}} + C_2 && \text{(Poincare inequality)} \\ &\geq C_4 \|f\|_{W^{1,p}}^p + C_5. && \text{(Since } p > 1\text{)} \end{aligned}$$

Hence $E[f]$ is bounded below. Consider an infimising sequence f_k i.e. $E[f_k] \to \inf_{f \in \mathcal{A}} E[f]$. Using the above inequality,

$$E[f_k] \geq C_4 \|f_k\|_{W^{1,p}}^p + C_5.$$

Since $E[f_k]$ is a convergent sequence, it is bounded, therefore we have a uniform bound on $\|f_k\|_{W^{1,p}}$. Since $W^{1,p}(0,1)$ is a reflexive Banach space for $p > 1$, we use Proposition 1 to conclude that f_k has a weakly convergent subsequence i.e., $f_k \rightharpoonup f_0$ weakly in $W^{1,p}(0,1)$. In particular this means that $f_k \rightharpoonup f_0$ in $L^p(0,1)$ and $F_k = f_k' \rightharpoonup f_0' = F_0$ in $L^p(0,1)$.

It is straightforward to see that $F_k \rightharpoonup F_0$ in $L^p(0,1)$ implies $F_k \to F_0$ in $L^1(0,1)$. From Theorem 2 we know that the first term in E is weakly lower semi-continuous with respect to $L^1(0,1)$ and since it only depends on F (and not f), we can deduce that it is weakly lower semi-continuous in $W^{1,p}(0,1)$. The second term as we have shown is a bounded linear functional on $W^{1,p}(0,1)$ so it is by construction weakly continuous. Hence $E[f]$ is sequentially weak semi-continuous on $W^{1,p}(0,1)$.

From the earlier discussion on the direct method, we thus have a candidate minimiser. All that is left is to show that $f_0 \in \mathcal{A}$. Since $W \to \infty$ as $F \to 0$, it is clear that F_0 cannot be 0 on set of positive measure as this would imply the minimum energy would be infinite (and we made the assumption that $\inf_{f \in \mathcal{A}} E[f] < \infty$). It is also not possible that $F_0 < 0$ on a positive measure set. This can be shown by contradiction: Suppose $F_0 < 0$ on $M \subset (0,1)$ $\mu(M) > 0$. We have from weak convergence, for all $g \in L^q(0,1)$,

$$\int_0^1 gF_k \,\mathrm{d}X \to \int_0^1 gF_0 \,\mathrm{d}X.$$

Let $g = \chi_M$ (the characteristic function on M), then since $F_k \geq 0$ a.e., $\int_0^1 gF_k \,\mathrm{d}X \geq 0$ for all k. However, $\int_0^1 gF_0 \,\mathrm{d}X < 0$, which is a contradiction.

Finally we would like to show that f_0 saatisfies the boundary conditions. Consider $w \in \mathcal{A}$, then $f_k - w \in W_0^{1,p}(0,1)$, which is a closed linear subspace so from Theorem 1 it is weakly closed. hence $f_0 - w \in W_0^{1,p}(0,1)$ so $f_0(0) = a$ and $f_0(1) = b$.

We conclude that $f_0 \in \mathcal{A}$ and it minimises E. □

6 Conclusions

The minimiser we have obtained in the above section is an element of the Sobolev space, $W^{1,p}$ with $p > 1$. Using the Sobolev embedding theorem, we can say that the minimiser is continuous, but nothing more can be said about its regularity in general. With some additional assumptions on W, however, it is possible to show that the minimiser is smooth [1]. In this case, it would also be a solution to the equilibrium equations (Euler-Lagrange equations). However, this problem is still open in the 3d case.

References

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