Ricci flow on surfaces: A priori bounds for curvature

Gokul Nair

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Consider a closed 2-manifold $\mathcal{M}$. The normalised Ricci flow is the following PDE:

$$\frac{\partial g}{\partial t} = (r - \mathrm{R})g. \tag{1}$$

where $g(t)$ is a 1-parameter family of smooth Riemannian metrics, $\mathrm{R}(x,t)$ is scalar curvature corresponding to the metric $g(t)$ and r is the average scalar curvature:

$$r = \frac{\int_{\mathcal{M}} \mathrm{R}\,\mathrm{d}x}{\int_{\mathcal{M}} \mathrm{d}x}.$$

It is straightforward to see that this flow preserves conformal class of the metric as it preserves right angles: suppose $u, v \in T_p\mathcal{M}$ such that $g(u,v) = 0$, then

$$\frac{\partial}{\partial t} g(u,v) = (r - \mathrm{R})g(u,v) = 0.$$

We prove that the curvature, R of a closed manifold with $r < 0$ subject to the normalised Ricci flow satisfies certain a priori bounds. First we compute the evolution equation for the curvature under the normalised Ricci flow:

Lemma 1. *Under the normalised Ricci flow on a surface, the scalar curvature R evolves according to the PDE:*

$$\frac{\partial \mathrm{R}}{\partial t} = \Delta_{g(t)}\mathrm{R} + \mathrm{R}(\mathrm{R} - r), \tag{2}$$

where Δ_g is the Laplace-Beltrami operator with respect to the metric $g(t)$.

Proof. Since the normalised Ricci flow preserves conformal class, we may write $g(t) = \mathrm{e}^{u(x,t)}h$ for a fixed metric, h. The curvatures of conformally related metrics are related by $\mathrm{R} = \mathrm{e}^{-u}(-\Delta_h u + \mathrm{R}_h)$, where Δ_h is the Laplacian w.r.t h [1, 12].
Since $\frac{\partial g}{\partial t} = (r - \mathrm{R})g$, we get $\frac{\partial u}{\partial t} = r - R$. Differentiating the expression for R,

$$\frac{\partial}{\partial t}\mathrm{R} = -\left(\frac{\partial u}{\partial t}\right)\mathrm{e}^{-u}(-\Delta_h u + \mathrm{R}_h) - \mathrm{e}^{-u}\Delta_h\left(\frac{\partial u}{\partial t}\right) = \Delta_g \mathrm{R} + \mathrm{R}(\mathrm{R} - r).$$

□

Equations of the type (2) are called Reaction-Diffusion equations. We first prove the maximum principle for such PDEs and use it obtain lower bounds on the curvature R. Upper bounds, on the other hand are more elusive. The strategy in this case is to study certain quantities related to special solutions to the normalised Ricci flow, known as *Ricci solitons*.

1 Maximum Principles for Reaction-Diffusion systems

We state and prove the maximum principle for scalar heat-type equations with non-linear reaction terms. Let $\mathcal{M}$ be a closed manifold and $v : \mathcal{M} \times [0,T)$ be a C^2 function. We first prove maximum principles for the following heat-type equation on a closed manifold:

$$\frac{\partial v}{\partial t} = \Delta_{g(t)} v + \langle X, \nabla v \rangle \,, \tag{3}$$

where $g(t)$ is a 1-parameter family of Riemannian metrics and $X(t)$ a 1-parameter family of smooth vector fields all defined on the interval $t \in [0,T)$. $\Delta_{g(t)}$ is the Laplace-Beltrami operator corresponding to the metric $g(t)$.

Theorem 2. *Let $u : \mathcal{M} \times [0,T) \to \mathbb{R}$ a C^2 function and $\exists \alpha \in \mathbb{R}$ such that $u(x,0) \geq \alpha$ for all $x \in \mathcal{M}$. If u is a super-solution to the heat-type equation, i.e.*

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + \langle X, \nabla u \rangle \,,$$

for all $(x,t) \in \mathcal{M} \times [0,T)$ such that $u(x,t) < \alpha$, then $u(x,t) \geq \alpha$ for all $(x,t) \in \mathcal{M} \times [0,T)$.

Proof. Consider the C^2 function $H : \mathcal{M} \times [0,T) \to \mathbb{R}$ defined by $H(x,t) := [u(x,t) - \alpha] + \epsilon t + \epsilon$ where $\epsilon > 0$. Let $(x_0, t_0) \in \mathcal{M} \times [0,T)$ be the point in spacetime where H attains its maximum among all points and all previous times i.e.

$$H(x_0,t_0) = \min_{\mathcal{M} \times [0,t_0]} H.$$

This exists because $\mathcal{M} \times [0,t_0]$ is compact. At (x_0,t_0) we have

$$\frac{\partial H}{\partial t} \leq 0; \quad \nabla H = 0; \quad \Delta_g H \geq 0. \tag{$*$}$$

Since $u(x,0) \geq \alpha$, we see that $H(x,0) \geq \epsilon > 0$.
Now we use the fact that u is a super-solution, whenever $u < \alpha$:

$$\begin{aligned} &\frac{\partial u}{\partial t} \geq \Delta_g u + \langle X, \nabla u \rangle \\ \implies &\frac{\partial H}{\partial t} \geq \Delta_g H + \langle X, \nabla H \rangle + \epsilon. \end{aligned} \tag{$**$}$$

Notice that

$$H > 0 \Leftrightarrow [u(x,t) - \alpha] + \epsilon t + \epsilon,$$

and since this is true for any $\epsilon > 0$ and $t \in [0,T)$, we have $u(x,t) \geq \alpha$. So we need only show that $H > 0$ for all $t \in [0,T)$.

Suppose $H \leq 0$ at some $(x_1,t_1) \in \mathcal{M} \times [0,T)$. Since $\mathcal{M}$ is compact and $H > 0$ at $t = 0$, there must be some first time $t_0 \in (0,t_1]$ such that at a point $x_0 \in \mathcal{M}$ such that $H(x_0,t_0) = 0$. Then since

$$u(x_0,t_0) = \alpha - \epsilon t_0 - \epsilon < \alpha,$$

using $(*)$ and $(**)$ we have

$$0 \geq \frac{\partial H}{\partial t}(x_0, t_0) \geq \Delta_g H(x_0, t_0) + \langle X, \nabla H \rangle (x_0, t_0) + \epsilon \geq \epsilon > 0.$$

This is a contradiction, so $H > 0$. □

Next we prove the maximum principle for heat-type equations with linear reaction terms. Let $\beta : \mathcal{M} \times [0, T) \to \mathbb{R}$ be a function such that for each $\tau \in [0, T)$, there exists a constant $C_\tau < \infty$ such that $\beta(x, t) \leq C_\tau$ for all $x \in \mathcal{M}$ and $t \in [0, \tau]$. We then consider the following heat-type equation:

$$\frac{\partial v}{\partial t} = \Delta_{g(t)} v + \langle X, \nabla v \rangle + \beta v. \tag{4}$$

Proposition 3. *Let $u : \mathcal{M} \times [0, T) \to \mathbb{R}$ be a C^2 super-solution to (4) on a closed manifold i.e.*

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + \langle X, \nabla u \rangle + \beta u.$$

If $u(x, 0) \geq 0$ for all $x \in \mathcal{M}$, then $u(x, t) \geq 0$ for all $(x, t) \in \mathcal{M} \times [0, T)$.

Proof. Given $\tau \in (0, T)$ define the auxiliary function

$$J(x, t) := \mathrm{e}^{-C_\tau t} u(x, t).$$

Since $u(x, 0) \geq 0$, we also have $J(x, 0) \geq 0$ for all $x \in \mathcal{M}$. Differentiating J:

$$\begin{aligned} \frac{\partial J}{\partial t} &= -C_\tau \mathrm{e}^{-C_\tau t} u(x, t) + \mathrm{e}^{-C_\tau t} \frac{\partial u}{\partial t} \\ &\geq -C_\tau J(x, t) + \mathrm{e}^{-C_\tau t} (\Delta_g u + \langle X, \nabla u \rangle + \beta u) \\ &= \Delta_g J + \langle X, \nabla J \rangle + \underbrace{(\beta - C_\tau)}_{\leq 0} J \end{aligned}$$

Thus for all $(x, t) \in \mathcal{M} \times [0, \tau]$, where $J \leq 0$ we have

$$\frac{\partial J}{\partial t} \geq \Delta_g J + \langle X, \nabla J \rangle .$$

Applying Theorem 2 we conclude that $J \geq 0$ for all $(x, t) \in \mathcal{M} \times [0, \tau)$, consequently $u \geq 0$ also. Since τ was arbitrary, the result follows. □

Next we prove the maximum principle for Reaction-Diffusion equations with nonlinear reaction terms. This theorem is often referred to as *parabolic maximum principle.*

$$\frac{\partial v}{\partial t} = \Delta_{g(t)} v + \langle X, \nabla v \rangle + F(v) \tag{5}$$

where $g(t)$ is a smooth 1-parameter family of metrics, for $t \in [0, T)$. $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function.

Definition 4. *A smooth function $u : \mathcal{M} \times [0,T) \to \mathbb{R}$ is a supersolution of* (5) *if*

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + F(u)$$

and a subsolution if

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + F(u).$$

Theorem 5 (Parabolic Maximum principle)**.** *Let $u : \mathcal{M} \times [0,T) \to \mathbb{R}$ be a C^2 supersolution to* (5) *on a closed manifold. Suppose $\exists C_1 \in \mathbb{R}$ such that $u(x,0) \geq C_1$ for all $x \in \mathcal{M}$ and let φ_1 be a solution to the ODE initial value problem*

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = F(\varphi_1) \qquad \varphi_1(0) = C_1,$$

then

$$u(x,t) \geq \varphi_1(t)$$

for all $x \in \mathcal{M}$ and $t \in [0,T)$ such that $\varphi_1(t)$ exists.

Similarly suppose $u : \mathcal{M} \times [0,T) \to \mathbb{R}$ is a C^2 subsolution to (5) *and $\exists C_2 \in \mathbb{R}$ such that $u(x,0) \leq C_2$ for all $x \in \mathcal{M}$ and let φ_2 be a solution to the ODE initial value problem*

$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = F(\varphi_2) \qquad \varphi_2(0) = C_2,$$

then

$$u(x,t) \leq \varphi_2(t)$$

for all $x \in \mathcal{M}$ and $t \in [0,T)$ such that $\varphi_2(t)$ exists.

Proof. First we prove the lower bound. The upper bound follows similarly.

$$\begin{aligned}\frac{\partial}{\partial t}(u - \varphi_1) &= \frac{\partial u}{\partial t} - \frac{\partial \varphi_1}{\partial t} \\ &\geq \Delta_g u + \langle X, \nabla u \rangle + F(u) - F(\varphi_1) \\ &= \Delta_g (u - \varphi_1) + \langle X, \nabla (u - \varphi_1) \rangle + F(u) - F(\varphi_1).\end{aligned}$$

From the assumption on initial date $u - \varphi_1 \geq 0$ at $t = 0$. Pick $\tau \in (0,T)$. Since $\mathcal{M}$ is compact, there exists $C_\tau < \infty$ such that for all $(x,t) \in \mathcal{M} \times [0,\tau]$

$$|u(x,t)| \leq C_\tau \quad |\varphi_1(t)| \leq C_\tau.$$

Since F is locally Lipschitz, there exists Ł$_\tau < \infty$ such that

$$|F(v) - F(w)| \leq L_\tau \, |v - w|$$

for all $v, w \in [-C_\tau, C_\tau]$. Hence

$$-L_\tau \, |v - w| \leq F(v) - F(w) \leq L_\tau \, |v - w| \, ,$$

$$-L_\tau \operatorname{sgn}(v-w)\cdot(v-w) \le F(v)-F(w) \le L_\tau \operatorname{sgn}(v-w).$$

Using the above inequality:

$$\frac{\partial u - \varphi_1}{\partial t} \ge \Delta_g(u-\varphi_1) + \langle X, \nabla(u-\varphi_1)\rangle - L_\tau \operatorname{sgn}(u-\varphi_1)\cdot(u-\varphi_1)$$

Let $\beta := -L_\tau \operatorname{sgn}(u-\varphi_1)$ and use Theorem 3 we may conclude that

$$u - \varphi_1 \ge 0$$

on $\mathcal{M}\times[0,\tau]$. The theorem follows since $\tau\in(0,T)$ was arbitrary. □

2 Lower Bounds for Curvature

We return to the evolution equation for scalar curvature under the normalised Ricci flow,

$$\frac{\partial \mathrm{R}}{\partial t} = \Delta_{g(t)}\mathrm{R} + \mathrm{R}(\mathrm{R}-r).$$

It is clear that this is an equation of the type (5) and thus we may apply the maximum principle to get the following estimate

Lemma 6. *Let $g(t)$ be a solution with bounded curvature of the normalised Ricci flow on a closed surface with $r<0$. Then*

$$\mathrm{R}-r \ge \frac{r}{1-\left(1-\frac{r}{\mathrm{R}_{\min}(0)}\right)\mathrm{e}^{rt}} - r \ge (\mathrm{R}_{\min}-r)\mathrm{e}^{rt},$$

where $\mathrm{R}_{\min}(t) := \inf_{x\in\mathcal{M}} \mathrm{R}(x,t)$.

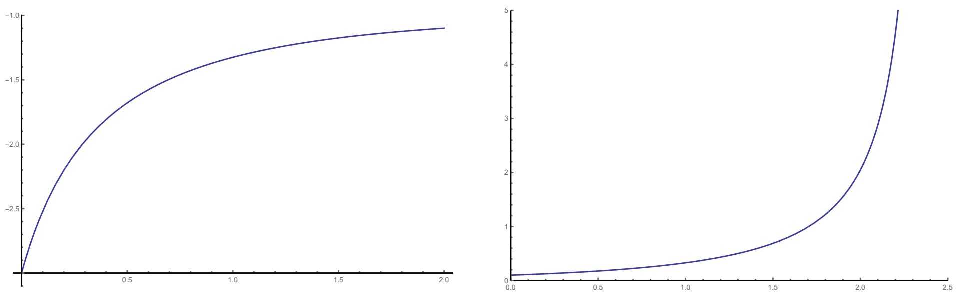
Proof. We apply the maximum principle to the solution of (2). Formally ignoring the Laplacian term gives us an ODE of the form $\mathrm{d}s/\mathrm{d}t = s(s-r)$. Solving this with the initial condition $s(0)=s_0$ with $r<0$ yields:

$$s(t) = \frac{r}{1-(1-r/s_0)\,\mathrm{e}^{rt}}. \tag{6}$$

Next we would like a uniform lower bound on $\mathrm{R}(x,0)$ as the initial condition for the ODE. We pick $\mathrm{R}_{\min}(0)$, which exists because the manifold is compact. Applying the maximum principle with $s_0 = \mathrm{R}_{\min}(0)$ we get

$$\mathrm{R} \ge \frac{r}{1-\left(1-\frac{r}{\mathrm{R}_{\min}(0)}\right)\mathrm{e}^{rt}}.$$

The inequality in the lemma is straightforward to show now. □



(a) $s_0 < 0$. (b) $s_0 > 0$.

Figure 1: $s(t)$, from equation (6) plotted for negative and positive values of initial condition.

We could also apply the maximum principle to obtain a similar upper bound for R, however these are not particularly useful. This is because $s(t)$ in (6) blows up to infinity at

$$t_b = -\frac{1}{r} \log \left(1 - \frac{r}{s_0} \right).$$

When $s_0 < 0$ (remembering that $r < 0$ in our case), $t_b < 0$ and this does not affect the flow for positive time. However, when $s_0 > 0$, $0 < t_b < \infty$ i.e. the solution blows up at finite time. To obtain lower bounds we use $s_0 = \mathrm{R}_{\min}(0)$, which is guaranteed to be negative since the average r is negative. On the other hand, to obtain upper bounds we have no guarantee that there exists a negative upper-bound on $\mathrm{R}(x, 0)$ as there could be regions of the manifold with positive curvature, despite the average being negative. Thus there is the possibility that upper-bounds we obtain for curvature could blow up in finite time, rendering them useless.

3 Upper bounds for curvature

In the earlier section we saw that applying the maximum principle directly to the Reaction-Diffusion equation for curvature does not, in general, give useful upper-bounds, so we are forced to look for other geometric quantities that will help us in our cause:

Definition 7 (Curvature Potential)**.** *A function $f(x, t)$ satisfying the following Poisson equation,*

$$\Delta_g f = \mathrm{R} - r \tag{7}$$

is called a potential of the curvature function.

We notice that the above equation has a solution as $\int_{\mathcal{M}} \mathrm{R} - r \, \mathrm{d}\mu = 0$, which is a necessary and sufficient condition for the Poisson equation to be soluble on a closed Riemannian manifold. The solution is unique up to addition with a function constant in space $c(t)$. It should be mentioned that the motivation for defining the potential as above comes from the study of a special class of solutions to the Ricci flow called Ricci solitons. Studying how f evolves under the Ricci normalised Ricci flow leads us to certain quantities that help us estimate the curvature from above.

Lemma 8. *Let $g(t)$ be a solution to the normalised Ricci flow on a closed surface $\mathcal{M}$. Then there is a corresponding potential function $f(x,t)$ that satisfies the evolution equation*

$$\frac{\partial f}{\partial t} = \Delta_g f + rf. \tag{8}$$

Proof. Let $f(x,t) = f_0(x,t) + c(t)$, where $f_0(x,t)$ is a potential of curvature and $c(t)$ is a function independent of space. We start by differentiating the equation $\Delta_g f = \mathrm{R} - r$ with respect to time and use the formula $\frac{\partial}{\partial t}\Delta_{g(t)} = (\mathrm{R}-r)\Delta_{g(t)}$. The left hand side yields,

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_g f) &= \left(\frac{\partial}{\partial t}\Delta_g\right) f + \Delta_g \frac{\partial f}{\partial t} \\ &= (\mathrm{R}-r)\Delta_g f + \Delta_g\left(\frac{\partial f_0}{\partial t} + \frac{\partial c}{\partial t}\right) \\ &= (\mathrm{R}-r)^2 + \Delta_{g(t)}\left(\frac{\partial f_0}{\partial t}\right).\end{aligned}$$

The right hand side,

$$\begin{aligned}\frac{\partial \mathrm{R}}{\partial t} &= \Delta_g \mathrm{R} + \mathrm{R}(\mathrm{R}-r) \quad \text{Using (2)} \\ &= \Delta_g^2 f_0 + \mathrm{R}(\mathrm{R}-r),\end{aligned}$$

where $\Delta_g^2 = \Delta_g \circ \Delta_g$ is the biharmonic operator on the manifold. Equating the two sides we get

$$\begin{aligned}\Delta_{g(t)}\left(\frac{\partial f_0}{\partial t}\right) &= \Delta_g^2 f_0 + r(\mathrm{R}-r) \\ &= \Delta_g(\Delta_g f_0 + r f_0)\end{aligned}$$

Thus,

$$\frac{\partial f_0}{\partial t} = \Delta_g f_0 + r f_0 + \gamma(t),$$

where $\gamma(t)$ is a a constant function of space. If we choose $c(t) = -\mathrm{e}^{rt}\int_0^t \mathrm{e}^{-rs}\gamma(s)\,\mathrm{d}s$, we are done. □

Finally we define the following quantity that will give us the required upper bound.

$$H := \mathrm{R} - r + |\nabla f|^2 \tag{9}$$

It is not straightforward to assign any geometric meaning to H apart from being a manufactured function that is a means to an end. We look at the evolution equation for H under the normalised Ricci flow:

Proposition 9. *Under the normalised Ricci flow the quantity H defined in (9) evolves according to the following PDE:*

$$\frac{\partial H}{\partial t} = \Delta_g H - 2|M|^2 + rH, \tag{10}$$

where M is the traceless part of the Hessian of f:

$$M := \nabla^2 f - \left(\frac{1}{2}\Delta_g f\right) g.$$

Proof. Notice first that we can manipulate (2) and use (7) to get

$$\begin{aligned}\frac{\partial}{\partial t}(\mathrm{R} - r) &= \Delta_g \mathrm{R} + \mathrm{R}(\mathrm{R} - r)\\ &= \Delta_g(\mathrm{R} - r) + (\mathrm{R} - r)^2 + r(\mathrm{R} - r)\\ &= \Delta_g(\mathrm{R} - r) + (\Delta_g f)^2 + r(\mathrm{R} - r). \end{aligned} \tag{†}$$

Now let us compute the evolution equation for $|\nabla f|^2$. We will use coordinates in this computation. We denote the components of the inverse of the metric by g^{ij}. It is straightforward to show that under the normalised Ricci flow, the inverse metric evolves according to

$$\frac{\partial g^{ij}}{\partial t} = (R - r)g^{ij} \tag{11}$$

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f|^2 &= \frac{\partial}{\partial t}(g^{ij}\nabla_i f \nabla_j f)\\ &= \left(\frac{\partial g^{ij}}{\partial t}\right)\nabla_i f \nabla_j f + 2g^{ij}\left(\frac{\partial}{\partial t}\nabla_i f\right)\nabla_j f\\ &= (\mathrm{R} - r)g^{ij}\nabla_i f \nabla_j f + 2g(\nabla \Delta_g f + r\nabla f, \nabla f) \quad \text{Using (11) and (8)}\\ &= (\mathrm{R} - r)|\nabla f|^2 + g(\Delta_g \nabla f, \nabla f) + 2r|\nabla f|^2\\ &= \mathrm{R}|\nabla f|^2 + r|\nabla f|^2 + g(\Delta_g \nabla f, \nabla f)\end{aligned}$$

In the above equation we denote the inner product on the cotangent space, $T_p^*\mathcal{M}$ as $g(\cdot,\cdot)$. We now need a formula from Riemannian geometry to commute the Laplacian and the gradient:

$$\nabla \Delta_g = \Delta_g \nabla - \frac{1}{2}\mathrm{R}\nabla \tag{12}$$

Using this formula in the our earlier computation we get

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f|^2 &= r|\nabla f|^2 + 2g(\Delta_g \nabla f, \nabla f)\\ &= r|\nabla f|^2 + \Delta_g|\nabla f|^2 - 2\left|\nabla^2 f\right|^2.\end{aligned} \tag{††}$$

Combining (†) with (††) yields the result because

$$|M|^2 = \left|\nabla^2 f\right|^2 - \frac{1}{2}(\Delta_g f)^2.$$

□

It is clear from the definition (9) that $H > \mathrm{R} - r$. Since H is a solution to (10), it is a sub-solution to the PDE $\dfrac{\partial v}{\partial t} = \Delta_g v + rv$ since

$$\frac{\partial H}{\partial t} = \Delta_g H - 2\left|M\right|^2 + rH \leq \Delta_g H + rH \tag{13}$$

To apply the maximum principle to (13) we formally neglect the Laplacian and solve the resulting ODE $\mathrm{d}v/\,\mathrm{d}t = rv$ to get the following estimate:

$$R - r \leq H \leq C\mathrm{e}^{rt}.$$

Combining the above estimate with Lemma 6 we may conclude

Proposition 10. *For a solution of the normalised Ricci flow on a closed surface $\mathcal{M}$ of negative average scalar curvature ($r < 0$), $\exists C > 0$ such that*

$$r - C\mathrm{e}^{rt} \leq R \leq r + C\mathrm{e}^{rt}.$$

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