Proof of the Uniformisation theorem using Ricci Flow

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The Uniformisation theorem

Theorem (Uniformisation)

Let $\mathcal{M}$ be a closed Riemann surface. Then $\mathcal{M}$ is conformally equivalent to one of these constant curvature surfaces:

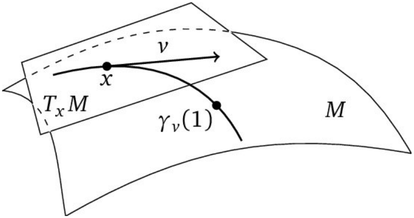
(i) *the Unit disk* $(\mathbb{D})$ *(curvature -1),*

(ii) *the Complex plane* $(\mathbb{C})$ *(curvature 0),*

(iii) *or the Riemann sphere* $(\mathbb{C} \cup \{\infty\})$ *(curvature +1).*

In this presentation: We will prove a (i) of this theorem for closed 2-manifolds using the Ricci flow.

Ricci curvature



- $\exp : T_x\mathcal{M} \to \mathcal{M}$ gives us a way to map a ball in $T_x\mathcal{M} \cong \mathbb{R}^n$ into $\mathcal{M}$.
- The volume of this image can be Taylor expanded in *normal coordinates* as follows:

$$\mathrm{d}\mu_{\mathcal{M}}(x) = (1 - \frac{1}{6}R_{ij}|_x x^i x^j + \ldots)\mathrm{d}\mu_{\text{Euclidean}}$$

- The Ricci curvature at a point x is a $(2, 0)$-tensor with components R_{ij}.

‘Basic’ facts

- Rc is a differential operator on the space of metrics.
- The scalar curvature is the trace of the Ricci curvature,

$$\mathrm{R} = \mathrm{tr}_g(\mathrm{Rc}) = g^{ij}R_{ij}$$

- 2d manifolds satisfy the Einstein equation:

$$\mathrm{Rc} = \frac{1}{2}\mathrm{R}g$$

- The average scalar curvature is defined as

$$r = \frac{\int_{\mathcal{M}} \mathrm{R}\,\mathrm{d}\mu}{\int_{\mathcal{M}} \mathrm{d}\mu}.$$

‘Basic’ facts

(a) $r > 0$

(b) $r = 0$

(c) $r < 0$

Ricci flow

The Ricci flow is the PDE:

$$\frac{\partial g}{\partial t} = -2\mathrm{Rc} \qquad g(0) = g_0$$

The volume of $\mathcal{M}$ may change under the Ricci flow so we define the normalised Ricci flow:

$$\frac{\partial g}{\partial t} = -2\mathrm{Rc} + \frac{2}{n} r g \qquad g(0) = g_0.$$

Using the Einstein equation, for surfaces:

$$\frac{\partial g}{\partial t} = (r - \mathrm{R}) g \qquad g(0) = g_0.$$

r is a constant by virtue of the Gauss-Bonnet theorem.

Ricci flow

Our main result is the following:

Theorem

Let $(\mathcal{M}, g_0)$ *is a closed Riemannian surface with average scalar curvature,* $r < 0$*. Then there exists a unique solution* $g(t)$ *to the normalised Ricci flow*

$$\frac{\partial g}{\partial t} = (r - \mathrm{R})g \qquad g(0) = g_0.$$

The solution exists for all time and as $t \to \infty$, $g(t)$ *converges exponentially in every* C^k*-norm to a smooth constant-curvature metric* g_∞*.*

Short-time existence

- The Ricci flow is a non-linear PDE in the space of metrics.
- In fact we can show that in a harmonic coordinate system, the Ricci curvature

$$R_{ij} = -\frac{1}{2}\Delta g_{ij} + Q_{ij}(g^{-1}, \partial g),$$

  where Δ is the Laplacian , and Q is a quadratic term.
- Thus it looks like non-linear heat equation in harmonic coordinates.
- However, as the metric evolves, the coordinate system will not remain Harmonic.
- DeTurck's trick: Introduce time dependent infinitesimal coordinate changes that keep the equation parabolic.

DeTurck's trick: some details

- For an appropriate choice of vector field $X(t)$, the following PDE is strongly parabolic:

$$\frac{\partial \bar{g}_{ij}(t)}{\partial t} = -2\bar{R}_{ij}(t) + (\mathcal{L}_{X(t)}\bar{g}(t))_{ij} \qquad \bar{g}(0) = g_0, \tag{1}$$

  where $\bar{R}_{ij}(t)$ are the components of $\mathrm{Rc}[\bar{g}(t)]$.
- Then, we construct a 1-parameter family of diffeomorphisms, $\{\varphi_t : \mathcal{M} \to \mathcal{M}\}$ by solving the ODE pointwise:

$$\frac{\partial \varphi_t(p)}{\partial t} = -X(\varphi_t(p), t) \qquad \varphi_0 = \mathrm{id}_{\mathcal{M}}.$$

- Then the family of metrics $g(t) := \varphi_t^* \bar{g}(t)$ solves the Ricci flow.

Short-time existence

With the appropriate choice: $X^k = g^{pq}(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq})$, where $\tilde{\Gamma}$ is the Levi-Civita connection of a fixed background metric, we get the following existence result:

Theorem (Hamilton-DeTurck)

If $(\mathcal{M}, g_0)$ is a closed Riemannian manifold, there exists a unique solution $g(t)$ to the Ricci flow defined on some time interval $[0, \varepsilon)$ $(\varepsilon > 0)$ such that $g(0) = g_0$.

Lower bounds for curvature

- From this point on, we consider dimension=2.
- recall that the Ricci flow is given by $\frac{\partial g}{\partial t} = (r - \mathrm{R})g$.
- Ricci flow preserves conformal class: Let $u, v \in T_p\mathcal{M}$ such that $g(u,v) = 0$ then,

$$\frac{\partial}{\partial t} g(u,v) = (r - \mathrm{R})g(u,v) = 0$$

Lemma

Under the normalised Ricci flow, $\frac{\partial g}{\partial t} = (r - \mathrm{R})g$ *the scalar curvature evolves as*

$$\frac{\partial \mathrm{R}}{\partial t} = \Delta \mathrm{R} + \mathrm{R}(\mathrm{R} - r).$$

Lower bounds for curvature

Proof:

Since the normalised Ricci flow preserves conformal class, we may write $g(t) = \mathrm{e}^{u(x,t)}h$ for a fixed metric, h. The curvatures of conformally related metrics are related by $\mathrm{R} = \mathrm{e}^{-u}(-\Delta_h u + \mathrm{R}_h)$, where Δ_h is the Laplacian w.r.t h.

Since $\dfrac{\partial g}{\partial t} = (r - \mathrm{R})g$, we get $\dfrac{\partial u}{\partial t} = r - R$. Differentiating the expression for R,

$$\frac{\partial}{\partial t}\mathrm{R} = -\left(\frac{\partial u}{\partial t}\right)\mathrm{e}^{-u}(-\Delta_h u + \mathrm{R}_h) - \mathrm{e}^{-u}\Delta_h\left(\frac{\partial u}{\partial t}\right) = \Delta\mathrm{R} + \mathrm{R}(\mathrm{R} - r)$$

Maximum principles

We state the maximum principle for semi-linear Reaction-Diffusion equations,

$$\frac{\partial v}{\partial t} = \Delta_{g(t)} v + \langle X(t), \nabla v \rangle + F(v) \tag{2}$$

where $g(t)$ and $X(t)$ are smooth 1-parameter families of metrics and vector fields for $t \in [0, T)$. $F : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function.

Definition

A smooth function $u : \mathcal{M} \times [0, T) \to \mathbb{R}$ is a super(sub)solution of (2) if

$$\frac{\partial u}{\partial t} \geq (\leq) \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + F(u)$$

Maximum principles

Theorem

Let $u : \mathcal{M} \times [0, T) \to \mathbb{R}$ *be a* C^2 *super-solution to* (2) *on a closed manifold. Suppose* $\exists C_1 \in \mathbb{R}$ *such that* $u(x, 0) \geq C_1$ *for all* $x \in \mathcal{M}$ *and let* φ_1 *be a solution to the ODE initial value problem*

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = F(\varphi_1) \qquad \varphi_1(0) = C_1,$$

then

$$u(x, t) \geq \varphi_1(t)$$

for all $x \in \mathcal{M}$ *and* $t \in [0, T)$ *such that* $\varphi_1(t)$ *exists.*

The same applies to subsolutions with the inequalities reversed.

Lower bounds for curvature

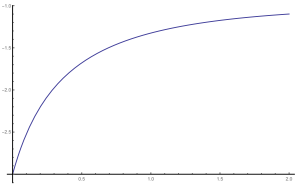
Let us apply the maximum principle to the equation

$$\frac{\partial \mathrm{R}}{\partial t} = \Delta \mathrm{R} + \mathrm{R}(\mathrm{R} - r).$$

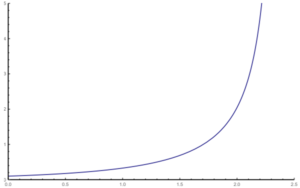
Formally ignoring the Laplacian and solving the ODE $\mathrm{d}s/\mathrm{d}t = s(s - r)$ with the initial condition $s(0) = s_0$, we get

$$s(t) = \frac{r}{1 - (1 - r/s_0)\,\mathrm{e}^{rt}}$$

Lower bounds for curvature



(a) ODE solution for $s_0 < 0$.



(b) ODE solution for $s_0 > 0$.

- Hence we could pick $s_0 = \mathrm{R}_{\min}(t) := \inf_{x \in \mathcal{M}} \mathrm{R}(x,t)$ to get a good lower bound when $r < 0$:

$$\mathrm{R} - r \geq \frac{r}{1 - (1 - r/\mathrm{R}_{min}(0))\, \mathrm{e}^{rt}} - r \geq C\mathrm{e}^{rt}.$$

- However the same would not work for upper-bounds since the ODE solution blows up in finite time for $s_0 > 0$.

Upper bounds

- Directly applying Maximum principle— BAD IDEA.
- Look for other quantities that can help us.
- ‘Ricci Solitons’: Source of inspiration.
- Self-similar solution or Ricci Soliton: $\exists \varphi_t$ conformal diffeomorphisms s.t.

$$g(t) = \varphi(t)^* g(0),$$

- Differentiating w.r.t. time this we get,

$$\frac{\partial g_{ij}}{\partial t} = (\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$$

- But g is also a solution to the normalised Ricci flow, so

$$(r - \mathrm{R}) g_{ij} = \nabla_i X_j + \nabla_j X_i$$

Curvature Potentials

- If there's a function $f(x,t)$ such that $X = -\nabla f$, then
$$(\mathrm{R} - r)g_{ij} = 2\nabla_i \nabla_j f.$$
- Tracing the above equation yields the Poisson equation for f:
$$\Delta f = \mathrm{R} - r.$$
- Key idea: The above equation is soluble even on non-solitons, because
$$\int_{\mathcal{M}} (\mathrm{R} - r)\mathrm{d}A = 0.$$
- We call such a function the *potential of curvature.*
- On closed surfaces, the potential is unique up to a constant $c(t)$.

Curvature Potentials

Lemma

Corresponding to a solution of the normalised Ricci flow, there is a potential evolving according to

$$\frac{\partial f}{\partial t} = \Delta f + rf. \tag{3}$$

We define the quantity,

$$H := \mathrm{R} - r + |\nabla f|^2 ,$$

which will be useful to get upper-bounds for curvature.
Clearly,

$$H \geq \mathrm{R} - r$$

Upper-bounds for Curvature

Lemma

For a solution of the normalised Ricci flow, H evolves according to the PDE

$$\frac{\partial H}{\partial t} = \Delta H - 2|M|^2 + rH, \tag{4}$$

where M is the traceless part of the Hessian of f,

$$M = \nabla^2 f - \left(\frac{1}{2}\Delta f\right) g$$

Clearly H is a sub-solution to the PDE $\frac{\partial v}{\partial t} = \Delta v + rv$ since

$$\frac{\partial H}{\partial t} = \Delta H - 2|M|^2 + rH \leq \Delta H + rH.$$

A priori Curvature Estimates

Applying the maximum principle to (4), we get

$$\mathrm{R} - r \leq H \leq C\mathrm{e}^{rt}.$$

Combining this with the lower bound we got earlier,

Proposition

For a solution of the normalised Ricci flow on a closed surface $\mathcal{M}$ of negative average scalar curvature $(r < 0)$, $\exists C > 0$ such that

$$r - C\mathrm{e}^{rt} \leq \mathrm{R} \leq r + C\mathrm{e}^{rt}.$$

Long-time existence

- Short-time existence.
- Bounds on curvature.
- Bernstein-Bando-Shi estimates $\Longrightarrow$ bounds on derivatives of curvature.
- As a consequence we can extend the lifetime of the solution to ∞ or ‘Long-time’ existence.
- In order to prove convergence of the solution to a constant curvature metric: suffice to show that all the derivatives of curvature die exponentially.

Convergence

Lemma

For any solution $g(t)$ of the normalised Ricci flow, the quantity $|\nabla R|^2$ evolves:

$$\frac{\partial |\nabla \mathrm{R}|^2}{\partial t} = \Delta |\nabla \mathrm{R}|^2 - 2\left|\nabla^2 \mathrm{R}\right|^2 + (4\mathrm{R} - 3r)|\nabla \mathrm{R}|^2$$

Proof:

We need the Ricci identity: $\nabla\Delta = \Delta\nabla - \frac{1}{2}\mathrm{R}\nabla$.

Recall the evolution equation for R, $\dfrac{\partial \mathrm{R}}{\partial t} = \Delta\mathrm{R} + \mathrm{R}(\mathrm{R} - r)$.

Using the above we get

$$\frac{\partial(\nabla \mathrm{R})}{\partial t} = \nabla(\Delta\mathrm{R} + \mathrm{R}(\mathrm{R} - r)) = \Delta\nabla\mathrm{R} + \frac{3}{2}R\nabla\mathrm{R} - r\nabla\mathrm{R}.$$

Convergence

Now we compute

$$\frac{\partial \left|\nabla \mathrm{R}\right|^2}{\partial t} = \frac{\partial}{\partial t} g(\nabla \mathrm{R}, \nabla \mathrm{R})$$
$$= \frac{\partial g}{\partial t}(\nabla \mathrm{R}, \nabla \mathrm{R}) + 2g\left(\frac{\partial \nabla \mathrm{R}}{\partial t}, \nabla \mathrm{R}\right)$$
$$= (\mathrm{R} - r)\left|\nabla \mathrm{R}\right|^2 + 2g(\Delta \nabla \mathrm{R} + \frac{3}{2} R \nabla \mathrm{R} - r \nabla \mathrm{R}, \nabla \mathrm{R})$$

Now, using the identity $\Delta \left|\nabla \mathrm{R}\right|^2 = 2g\left(\Delta \nabla \mathrm{R}, \nabla \mathrm{R}\right) + 2\left|\nabla^2 \mathrm{R}\right|^2$ (which is not very hard to show), we get the required equation. □

Convergence

Using our estimate for curvature $|\mathrm{R} - r| \leq C\mathrm{e}^{rt}$ in the above evolution equation:

$$\begin{aligned}
\frac{\partial |\nabla \mathrm{R}|^2}{\partial t} &= \Delta |\nabla \mathrm{R}|^2 - 2\left|\nabla^2 \mathrm{R}\right|^2 + (4\mathrm{R} - 3r)|\nabla \mathrm{R}|^2 \\
&\leq \Delta |\nabla \mathrm{R}|^2 + (r + 4C\mathrm{e}^{rt})|\nabla \mathrm{R}|^2 \\
&\leq \Delta |\nabla \mathrm{R}|^2 + \frac{r}{2}|\nabla \mathrm{R}|^2 \quad \text{for large enough } t > 0.
\end{aligned}$$

We may quite easily apply the maximum principle to the above sub-solution to get

$$|\nabla \mathrm{R}|^2 \leq C_1 \mathrm{e}^{rt/2}$$

This argument can be bootstrapped to get estimates on higher derivatives.

Note on $r = 0$ and $r > 0$ case

- The case $r = 0$ takes a little more work, but similar techniques as before work.
- The case of $r > 0$ is much trickier and requires different techniques.
- Such as estimates on 'surface entropy'.
- Chow and Knopf also use the Kazdan-Warner identity (which is proved using the Uniformisation theorem) in the proof. Consequently, this does not constitute an independent proof of the Uniformisation theorem.
- However, Cheng, Lu and Tian provided a proof that is independent of the Uniformisation theorem.

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