Proof of the Uniformisation theorem using Ricci Flow

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1 Introduction

One of the great triumphs of 19th century mathematics is the uniformisation theorem, which classifies all the possible types of simply-connected 2-dimensional manifolds (surfaces). A version of this theorem for closed two manifolds may be stated as follows [1]

Theorem 1 (Uniformisation)**.** *Let $\mathcal{M}$ be a closed Riemann surface. Then $\mathcal{M}$ can be given the geometry of one of these constant curvature surfaces:*

(i) the Unit disk ($\mathbb{D}$) *(curvature -1),*

(ii) the Complex plane ($\mathbb{C}$) *(curvature 0),*

(iii) or the Riemann sphere ($\mathbb{C} \cup \{\infty\}$) *(curvature +1).*

A natural successor to this theorem is Thurston's geometrisation conjecture, which classifies closed 3-dimensional manifolds. One of its many corollaries, namely the Poincaré conjecture has garnered much attention since being named a Millennium prize problem. In 1982, Richard Hamilton established a programme to tackle the problem, which was finally completed by Perelman in 2003, thereby resolving the conjecture for good. The workhorse of Hamilton's programme is the nonlinear PDE called the Ricci flow thus cements analytic techniques as a powerful tool in extracting deep topological results.

Henri Poincaré and Paul Koebe are credited with providing the first rigorous proofs of the Uniformisation theorem in the early 20^{th} century [2, 3]. Unsurprisingly, there have been several different proofs of the theorem since, including one that involves the Ricci flow, mainly due to Hamilton, Chow, Chen, Lu and Tian [4, 5]. They employ the flow to uniformise arbitrary given metrics on closed surfaces to one of constant curvature -1, 0 or +1.

In this document we present the first of three such cases. In particular, we will see that a metric of constant negative curvature exists for surfaces of negative Euler characteristic. The main reference for this proof is [1].

2 The Ricci Flow PDE

The Ricci flow is a partial differential evolution equation that may be imposed on the metric of a smooth Riemannian manifold. We start with some basic definitions/notations we will need [6, 7]:

Definition 2. *A Riemannian metric, g on a differentiable manifold $\mathcal{M}$ is given by a scalar product on each tangent space $T_p\mathcal{M}$ which depends smoothly on the base point p.*

In other words, the Riemannian metric is a symmetric, positive definite $(2,0)$-tensor field on the manifold i.e. $g \in C^\infty(S_2^+T^*\mathcal{M})$.

Definition 3. *A closed Riemannian manifold $(\mathcal{M}, g)$ is a compact manifold $\mathcal{M}$ with no boundary together with a Riemannian metric, g.*

In order to write down the Ricci flow PDE, we first need to define the Ricci curvature. The Ricci curvature is usually defined as the trace of the Riemann curvature tensor, but this definition requires crucially on the affine connection and the covariant derivative. We instead present another definition of the Ricci curvature through the exponential map:

Let $(\mathcal{M}, g)$ be Riemannian n-dimensional manifold. Then there is a unique (up to sign) n-form ω such that $\omega(E_1, ..., E_n) = \pm 1$ for every orthonormal frame $\{E_1, ..., E_n\}$. Let $X_1, ..., X_n \in C^\infty(T\mathcal{M})$ be vector fields that are orthonormal at the point p. Let $x^1, ..., x^n$ be the corresponding normal coordinate system. Denote $\omega_{1,...,n} = \omega(X_1, ..., X_n)$. Taylor expanding $\omega_{1,...,n}$ in a neighbourhood of p we get

$$\omega_{1,...,n} = 1 - \frac{1}{6} R_{ij}|_p x^i x^j + ...$$

This is exactly the deviation in the volume of unit Euclidean ball in a curved Riemannian manifold [8].

Definition 4. *The Ricci curvature of a Riemannian manifold $(\mathcal{M}, g)$,* Rc *is a symmetric $(2,0)$-tensor field whose components in normal coordinates at any point p are given by $R_{ij}|_p$.*

In other words, the Ricci curvature is the bilinear form corresponding to the second order correction term in the Taylor expansion of the volume form. It is important to note that $\mathrm{Rc} \in C^\infty(S_2T^*\mathcal{M})$. It is possible, although tedious to show that the expression for R_ij involves the metric, its inverse and its derivatives and can be thought of as a nonlinear differential operator on the space of metrics.

In addition to Rc we often deal with the scalar curvature.

Definition 5. *The Scalar curvature,* R *is the trace of* Rc *with respect to the metric, g i.e.,*

$$\mathrm{R} = g^{ij} R_{ij}.$$

Note that as in the above equation, we will use Einstein summation convention throughout this document.

We also have the following Proposition (stated without proof) from Riemannian geometry [6]:

Proposition 6 (Einstein's equation)**.** *For a 2-dimensional manifold, $\mathcal{M}$ we have*

$$\mathrm{Rc} = \frac{1}{2}\mathrm{R}g. \tag{1}$$

We also define the average scalar curvature of a manifold as

$$r = \frac{\int_{\mathcal{M}} \mathrm{R}\, \mathrm{d}\mu}{\int_{\mathcal{M}} \mathrm{d}\mu}. \tag{2}$$

Due to the Gauss-Bonnet theorem [6], r is determined by the Euler characteristic of the manifold, a topological invariant and is constant under geometric flows. Now we may define the Ricci flow.

Definition 7. *Given a smooth one parameter family of Riemannian manifolds $(\mathcal{M}, g(t))$ the Ricci flow is the PDE*

$$\frac{\partial g}{\partial t} = -2\mathrm{Rc} \qquad g(0) = g_0. \tag{3}$$

Since the volume of the manifold may change under the flow, we define the normalised Ricci flow which preserves volume,

$$\frac{\partial g}{\partial t} = -2\mathrm{Rc} + \frac{2}{n}rg \qquad g(0) = g_0. \tag{4}$$

At this point it is important to note why Ricci flow makes sense as a PDE. The Ricci curvature among among different notions of curvature has the property of being a symmetric bilinear form just like the Riemannian metric, thus they are comparable quantities and both sides of the Ricci flow equation belong to the same space, i.e., $C^\infty(S_2T^*\mathcal{M})$.

Using Proposition 6 we see that the normalised Ricci flow for surfaces simplifies to

$$\frac{\partial g}{\partial t} = (r - \mathrm{R})g. \tag{5}$$

Our main result will be the following

Theorem 8. *Let $(\mathcal{M}, g_0)$ is a closed Riemannian surface with average scalar curvature, $r < 0$. Then there exists a unique solution $g(t)$ to the normalised Ricci flow*

$$\frac{\partial g}{\partial t} = (r - \mathrm{R})g \qquad g(0) = g_0.$$

The solution exists for all time and as $t \to \infty$, $g(t)$ converges exponentially in every C^k-norm to a smooth constant-curvature metric g_∞.

There are several steps involved in proving this theorem, but in short they are the following: First we prove existence of a solution for a finite interval of time (short-time existence in the jargon). In order to extend the lifetime of existence to infinity (long-time existence), we need to prove certain a priori bounds on the curvature and its derivatives. Once this is established, we can show that the metric converges to well-defined limit and that all derivatives of the curvature converge to 0 exponentially.

3 Short-time existence and DeTurck's trick

In this section we discuss the existence of solutions to the Ricci flow for short-time.
The Ricci flow is a non-linear partial differential equation in g because the Ricci curvature is a non-linear partial differential operator on the metric,

$$\mathrm{Rc}_g \equiv \mathrm{Rc}(g) : C^\infty(S_2^+ T^*\mathcal{M}) \to C^\infty(S_2 T^*\mathcal{M})$$

In fact it may be shown that in a harmonic coordinate system

$$R_{ij} = -\frac{1}{2}\Delta g_{ij} + Q_{ij}(g^{-1}, \partial g)$$

where Δ is the Laplacian acting on the component functions g_ij and Q_ij is a non-linear term involving the inverse of the metric and its derivatives. Therefore the Ricci flow looks like a non-linear heat equation in this coordinate system for fixed time.
However, as g evolves under the Ricci flow, the coordinate system ceases to be harmonic. DeTurck devised a trick wherein one simultaneously applies a smoothly evolving family of diffeomorphisms to the manifold such that the harmonic coordinate system remains harmonic at all time.

The difficulty in solving the Ricci flow stems from the diffeomorphism equivariance of the Ricci operator. Due to this, Rc_g is only a weakly elliptic operator and consequently the Ricci flow is only weakly parabolic and standard parabolic existence theory cannot be applied. DeTurck showed that by adding the Lie derivative of the metric with respect to a vector field that in turn depends on the metric, one can make the equation strongly parabolic. Physicists often call this procedure 'gauge-fixing'.

We make the observation that the following flow is strongly parabolic for an appropriate

$$\frac{\partial \bar{g}_{ij}(t)}{\partial t} = -2\bar{R}_{ij}(t) + (\mathcal{L}_{X(t)}\bar{g}(t))_{ij} \qquad \bar{g}(0) = g_0, \tag{6}$$

where $\bar{R}_{ij}(t) = (\mathrm{Rc}[\bar{g}(t)])_{ij}$ and $X(t)$ is a smooth 1-parameter family of vector field. Then, we construct a 1-parameter family of diffeomorphisms, $\{\varphi_t : \mathcal{M} \to \mathcal{M}\}$ by solving the ODE pointwise:

$$\frac{\partial \varphi_t(p)}{\partial t} = -X(\varphi_t(p), t) \qquad \varphi_0 = \mathrm{id}_{\mathcal{M}}.$$

Then, the family of metrics given by $g(t) := \varphi_t^* \bar{g}(t)$ solves the Ricci flow.

To solve the strongly parabolic equation, (6) one first uses Schauder estimates for parabolic systems to show the existence of solutions to the linearised problem in an appropriate Hölder space. One then uses the inverse function theorem for Banach spaces to deduce local existence of solutions to the nonlinear problem [9, 10, 11].

Theorem 9 (Hamilton-DeTurck)**.** *If $(\mathcal{M}, g_0)$ is a closed Riemannian manifold, there exists a unique solution $g(t)$ to the Ricci flow defined on some time interval $[0, \varepsilon)$ $(\varepsilon > 0)$ such that $g(0) = g_0$.*

For a formal treatment of the non-ellipticity of the Ricci operator we are required to understand the theory of symbols of differential operators between vector bundles, however we will avoid it altogether in this document.

4 A priori estimates for Curvature

In order to prove Theorem 8 we need to show that the short-time solution from the earlier section makes sense for all time. In order to do so we show certain uniform bounds on the curvature of solutions to the normalised Ricci flow. Uniform lower bounds are obtained by a straightforward application of the maximum principle for parabolic PDEs, however upper-bounds are more elusive. The strategy in this case is to study certain quantities related to special solutions to the normalised Ricci flow, known as *Ricci solitons*.

We state the maximum principle for scalar heat-type equations with non-linear reaction terms. Let $\mathcal{M}$ be a smooth closed manifold and $v : \mathcal{M} \times [0, T) \to \mathbb{R}$. We then consider the following heat-type equation:

$$\frac{\partial v}{\partial t} = \Delta_{g(t)} v + F(v) \tag{7}$$

where $g(t)$ is a smooth 1-parameter family of metrics, for $t \in [0, T)$. $F : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function.

Definition 10. *A smooth function* $u : \mathcal{M} \times [0, T) \to \mathbb{R}$ *is a supersolution of* (7) *if*

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + F(u)$$

and a subsolution if

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + F(u).$$

Theorem 11 (Parabolic Maximum principle)**.** *Let* $u : \mathcal{M} \times [0, T) \to \mathbb{R}$ *be a* C^2 *supersolution to* (7) *on a closed manifold. Suppose* $\exists C_1 \in \mathbb{R}$ *such that* $u(x, 0) \geq C_1$ *for all* $x \in \mathcal{M}$ *and let* φ_1 *be a solution to the ODE initial value problem*

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = F(\varphi_1) \qquad \varphi_1(0) = C_1,$$

then

$$u(x, t) \geq \varphi_1(t)$$

for all $x \in \mathcal{M}$ *and* $t \in [0, T)$ *such that* $\varphi_1(t)$ *exists.*

Similarly suppose $u : \mathcal{M} \times [0, T) \to \mathbb{R}$ *is a* C^2 *subsolution to* (7) *and* $\exists C_2 \in \mathbb{R}$ *such that* $u(x, 0) \leq C_2$ *for all* $x \in \mathcal{M}$ *and let* φ_2 *be a solution to the ODE initial value problem*

$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = F(\varphi_2) \qquad \varphi_2(0) = C_2,$$

then

$$u(x, t) \leq \varphi_2(t)$$

for all $x \in \mathcal{M}$ *and* $t \in [0, T)$ *such that* $\varphi_2(t)$ *exists.*

4.1 Lower Bounds for Curvature

From this point on, we exclusively study the normalised Ricci flow in dimension 2, 5 with average scalar curvature, $r < 0$. It is straightforward to see that this flow preserves conformal class of the metric as it preserves right angles: suppose $u, v \in T_p\mathcal{M}$ such that $g(u,v) = 0$, then

$$\frac{\partial}{\partial t} g(u,v) = (r - \mathrm{R})g(u,v) = 0.$$

Lemma 12. *Under the normalised Ricci flow on a surface, the scalar curvature R evolves according to the PDE:*

$$\frac{\partial \mathrm{R}}{\partial t} = \Delta_{g(t)}\mathrm{R} + \mathrm{R}(\mathrm{R} - r), \tag{8}$$

where Δ_g is the Laplace-Beltrami operator with respect to the metric $g(t)$.

Proof. Since the normalised Ricci flow preserves conformal class, we may write $g(t) = \mathrm{e}^{u(x,t)}h$ for a fixed metric, h. The curvatures of conformally related metrics are related by $\mathrm{R} = \mathrm{e}^{-u}(-\Delta_h u + \mathrm{R}_h)$, where Δ_h is the Laplacian w.r.t h [1, 12].
Since $\dfrac{\partial g}{\partial t} = (r - \mathrm{R})g$, we get $\dfrac{\partial u}{\partial t} = r - R$. Differentiating the expression for R,

$$\frac{\partial}{\partial t}\mathrm{R} = -\left(\frac{\partial u}{\partial t}\right)\mathrm{e}^{-u}(-\Delta_h u + \mathrm{R}_h) - \mathrm{e}^{-u}\Delta_h\left(\frac{\partial u}{\partial t}\right) = \Delta_g\mathrm{R} + \mathrm{R}(\mathrm{R} - r)$$

□

It is clear that this is an equation of the type (7) and thus we may apply the maximum principle to get the following estimate

Lemma 13. *Let $g(t)$ be a solution with bounded curvature of the normalised Ricci flow on a closed surface with $r < 0$. Then*

$$\mathrm{R} - r \geq \frac{r}{1 - \left(1 - \frac{r}{\mathrm{R}_{\min}(0)}\right)\mathrm{e}^{rt}} - r \geq (\mathrm{R}_{\min} - r)\mathrm{e}^{rt},$$

where $\mathrm{R}_{\min}(t) := \inf_{x\in\mathcal{M}} \mathrm{R}(x,t)$.

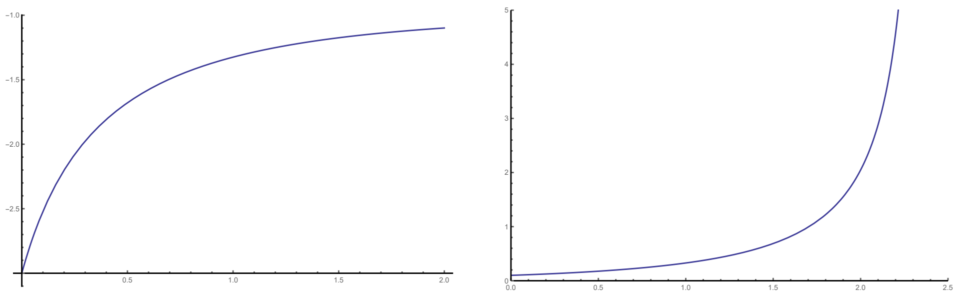
Proof. We apply the maximum principle to the solution of (8). Formally ignoring the Laplacian term gives us an ODE of the form $\mathrm{d}s/\mathrm{d}t = s(s - r)$. Solving this with the initial condition $s(0) = s_0$ with $r < 0$ yields:

$$s(t) = \frac{r}{1 - (1 - r/s_0)\,\mathrm{e}^{rt}}. \tag{9}$$

Next we would like a uniform lower bound on $\mathrm{R}(x,0)$ as the initial condition for the ODE. We pick $\mathrm{R}_{\min}(0)$, which exists because the manifold is compact. Applying the maximum principle with $s_0 = \mathrm{R}_{\min}(0)$ we get

$$\mathrm{R} \geq \frac{r}{1 - \left(1 - \frac{r}{\mathrm{R}_{\min}(0)}\right)\mathrm{e}^{rt}}.$$

The inequality in the lemma is straightforward to show now. □



(a) $s_0 < 0$.　(b) $s_0 > 0$.

Figure 1: $s(t)$, from equation (9) plotted for negative and positive values of initial condition.

We could also apply the maximum principle to obtain a similar upper bound for R, however these are not particularly useful. This is because $s(t)$ in (9) blows up to infinity at

$$t_b = -\frac{1}{r}\log\left(1 - \frac{r}{s_0}\right).$$

When $s_0 < 0$ (remembering that $r < 0$ in our case), $t_b < 0$ and this does not affect the flow for positive time. However, when $s_0 > 0$, $0 < t_b < \infty$ i.e. the solution blows up at finite time. To obtain lower bounds we use $s_0 = \mathrm{R}_{\min}(0)$, which is guaranteed to be negative since the average r is negative. On the other hand, to obtain upper bounds we have no guarantee that there exists a negative upper-bound on $\mathrm{R}(x, 0)$ as there could be regions of the manifold with positive curvature, despite the average being negative. Thus there is the possibility that upper-bounds we obtain for curvature could blow up in finite time, rendering them useless.

4.2 Upper bounds for curvature

In the earlier section we saw that applying the maximum principle directly to the Reaction-Diffusion equation for curvature does not, in general, give useful upper-bounds, so we are forced to look for other geometric quantities that will help us in our cause:

Definition 14 (Curvature Potential)**.** *A function $f(x,t)$ satisfying the following Poisson equation,*

$$\Delta_g f = \mathrm{R} - r \tag{10}$$

is called a potential of the curvature function.

We notice that the above equation has a solution as $\int_{\mathcal{M}} \mathrm{R} - r \, \mathrm{d}\mu = 0$, which is a necessary and sufficient condition for the Poisson equation to be soluble on a closed Riemannian manifold. The solution is unique up to addition with a function constant in space $c(t)$. It should be mentioned that the motivation for defining the potential as above comes from the study of a special class of solutions to the Ricci flow called Ricci solitons. Studying how f evolves under the Ricci normalised Ricci flow leads us to certain quantities that help us estimate the curvature from above.

Lemma 15. *Let $g(t)$ be a solution to the normalised Ricci flow on a closed surface $\mathcal{M}$. Then there is a corresponding potential function $f(x,t)$ that satisfies the evolution equation*

$$\frac{\partial f}{\partial t} = \Delta_g f + rf. \tag{11}$$

Proof. Let $f(x,t) = f_0(x,t) + c(t)$, where $f_0(x,t)$ is a potential of curvature and $c(t)$ is a function independent of space. We start by differentiating the equation $\Delta_g f = \mathrm{R} - r$ with respect to time and use the formula $\frac{\partial}{\partial t}\Delta_{g(t)} = (\mathrm{R} - r)\Delta_{g(t)}$. The left hand side yields,

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_g f) &= \left(\frac{\partial}{\partial t}\Delta_g\right) f + \Delta_g \frac{\partial f}{\partial t} \\ &= (\mathrm{R} - r)\Delta_g f + \Delta_g \left(\frac{\partial f_0}{\partial t} + \frac{\partial c}{\partial t}\right) \\ &= (\mathrm{R} - r)^2 + \Delta_{g(t)}\left(\frac{\partial f_0}{\partial t}\right).\end{aligned}$$

The right hand side,

$$\begin{aligned}\frac{\partial \mathrm{R}}{\partial t} &= \Delta_g \mathrm{R} + \mathrm{R}(\mathrm{R} - r) \quad \text{Using (8)} \\ &= \Delta_g^2 f_0 + \mathrm{R}(\mathrm{R} - r),\end{aligned}$$

where $\Delta_g^2 = \Delta_g \circ \Delta_g$ is the biharmonic operator on the manifold. Equating the two sides we get

$$\begin{aligned}\Delta_{g(t)}\left(\frac{\partial f_0}{\partial t}\right) &= \Delta_g^2 f_0 + r(\mathrm{R} - r) \\ &= \Delta_g(\Delta_g f_0 + r f_0)\end{aligned}$$

Thus,

$$\frac{\partial f_0}{\partial t} = \Delta_g f_0 + r f_0 + \gamma(t),$$

where $\gamma(t)$ is a a constant function of space. If we choose $c(t) = -\mathrm{e}^{rt}\int_0^t \mathrm{e}^{-rs}\gamma(s)\,\mathrm{d}s$, we are done. □

Finally we define the following quantity that will give us the required upper bound.

$$H := \mathrm{R} - r + |\nabla f|^2 \tag{12}$$

It is not straightforward to assign any geometric meaning to H apart from being a manufactured function that is a means to an end. We look at the evolution equation for H under the normalised Ricci flow:

Proposition 16. *Under the normalised Ricci flow the quantity H defined in (12) evolves according to the following PDE:*

$$\frac{\partial H}{\partial t} = \Delta_g H - 2|M|^2 + rH, \tag{13}$$

where M is the traceless part of the Hessian of f:

$$M := \nabla^2 f - \left(\frac{1}{2}\Delta_g f\right) g.$$

Proof. Notice first that we can manipulate (8) and use (10) to get

$$\begin{aligned}
\frac{\partial}{\partial t}(\mathrm{R} - r) &= \Delta_g \mathrm{R} + \mathrm{R}(\mathrm{R} - r) \\
&= \Delta_g(\mathrm{R} - r) + (\mathrm{R} - r)^2 + r(\mathrm{R} - r) \\
&= \Delta_g(\mathrm{R} - r) + (\Delta_g f)^2 + r(\mathrm{R} - r). \qquad (\dagger)
\end{aligned}$$

Now let us compute the evolution equation for $|\nabla f|^2$. We will use coordinates in this computation. We denote the components of the inverse of the metric by g^{ij}. It is straightforward to show that under the normalised Ricci flow, the inverse metric evolves according to

$$\frac{\partial g^{ij}}{\partial t} = (R - r) g^{ij} \qquad (14)$$

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla f|^2 &= \frac{\partial}{\partial t}(g^{ij}\nabla_i f \nabla_j f) \\
&= \left(\frac{\partial g^{ij}}{\partial t}\right)\nabla_i f \nabla_j f + 2g^{ij}\left(\frac{\partial}{\partial t}\nabla_i f\right)\nabla_j f \\
&= (\mathrm{R} - r) g^{ij}\nabla_i f \nabla_j f + 2g(\nabla \Delta_g f + r\nabla f, \nabla f) \quad \text{Using (14) and (11)} \\
&= (\mathrm{R} - r)|\nabla f|^2 + g(\Delta_g \nabla f, \nabla f) + 2r|\nabla f|^2 \\
&= \mathrm{R}|\nabla f|^2 + r|\nabla f|^2 + g(\Delta_g \nabla f, \nabla f)
\end{aligned}$$

In the above equation we denote the inner product on the cotangent space, $T_p^*\mathcal{M}$ as $g(\cdot,\cdot)$. We now need a formula from Riemannian geometry to commute the Laplacian and the gradient:

$$\nabla \Delta_g = \Delta_g \nabla - \frac{1}{2}\mathrm{R}\nabla \qquad (15)$$

Using this formula in the our earlier computation we get

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla f|^2 &= r|\nabla f|^2 + 2g(\Delta_g \nabla f, \nabla f) \\
&= r|\nabla f|^2 + \Delta_g |\nabla f|^2 - 2\left|\nabla^2 f\right|^2. \qquad (\dagger\dagger)
\end{aligned}$$

Combining (†) with (††) yields the result because

$$|M|^2 = \left|\nabla^2 f\right|^2 - \frac{1}{2}(\Delta_g f)^2.$$

□

It is clear from the definition (12) that $H > \mathrm{R} - r$. Since H is a solution to (13), it is a sub-solution to the PDE $\frac{\partial v}{\partial t} = \Delta_g v + rv$ since

$$\frac{\partial H}{\partial t} = \Delta_g H - 2\left|M\right|^2 + rH \leq \Delta_g H + rH \tag{16}$$

To apply the maximum principle to (16) we formally neglect the Laplacian and solve the resulting ODE $\mathrm{d}v/\,\mathrm{d}t = rv$ to get the following estimate:

$$R - r \leq H \leq C\mathrm{e}^{rt}.$$

Combining the above estimate with Lemma 13 we may conclude

Proposition 17. *For a solution of the normalised Ricci flow on a closed surface $\mathcal{M}$ of negative average scalar curvature ($r < 0$), $\exists C > 0$ such that*

$$r - C\mathrm{e}^{rt} \leq R \leq r + C\mathrm{e}^{rt}.$$

5 Long-time existence and Convergence

To show long-time existence we first need to establish bounds on all the derivatives of curvature. In order to prove Theorem 8 it is sufficient to show that all the derivatives of R converge to 0 exponentially. We first show that the first derivative $\left|\nabla R\right|^2$ is bounded.

Lemma 18. *For any solution $g(t)$ of the normalised Ricci flow, the quantity $\left|\nabla R\right|^2$ evolves:*

$$\frac{\partial\left|\nabla \mathrm{R}\right|^2}{\partial t} = \Delta_g \left|\nabla \mathrm{R}\right|^2 - 2\left|\nabla^2 \mathrm{R}\right|^2 + (4\mathrm{R} - 3r)\left|\nabla \mathrm{R}\right|^2$$

Proof. We again use the Ricci identity (15). Using this we get,

$$\frac{\partial(\nabla \mathrm{R})}{\partial t} = \nabla(\Delta_g \mathrm{R} + \mathrm{R}(\mathrm{R} - r)) = \Delta_g \nabla \mathrm{R} + \frac{3}{2} R \nabla \mathrm{R} - r \nabla \mathrm{R}.$$

We now compute

$$\begin{aligned}
\frac{\partial\left|\nabla \mathrm{R}\right|^2}{\partial t} &= \frac{\partial}{\partial t} g(\nabla \mathrm{R}, \nabla \mathrm{R}) \\
&= \frac{\partial g}{\partial t}(\nabla \mathrm{R}, \nabla \mathrm{R}) + 2g\left(\frac{\partial \nabla \mathrm{R}}{\partial t}, \nabla \mathrm{R}\right) \\
&= (\mathrm{R} - r)\left|\nabla \mathrm{R}\right|^2 + 2g(\Delta_g \nabla \mathrm{R} + \frac{3}{2} R \nabla \mathrm{R} - r \nabla \mathrm{R}, \nabla \mathrm{R})
\end{aligned}$$

Now, using the identity $\Delta_g \left|\nabla \mathrm{R}\right|^2 = 2g\left(\Delta_g \nabla \mathrm{R}, \nabla \mathrm{R}\right) + 2\left|\nabla^2 \mathrm{R}\right|^2$, we get the required equation. □

Using our estimate for curvature $|\mathrm{R} - r| \leq Ce^{rt}$ in the above evolution equation:

$$\begin{aligned} \frac{\partial |\nabla \mathrm{R}|^2}{\partial t} &= \Delta_g |\nabla \mathrm{R}|^2 - 2 \left|\nabla^2 \mathrm{R}\right|^2 + (4\mathrm{R} - 3r) |\nabla \mathrm{R}|^2 \\ &\leq \Delta_g |\nabla \mathrm{R}|^2 + (r + 4Ce^{rt}) |\nabla \mathrm{R}|^2 \\ &\leq \Delta_g |\nabla \mathrm{R}|^2 + \frac{r}{2} |\nabla \mathrm{R}|^2 \quad \text{for large enough } t > 0. \end{aligned}$$

We may quite easily apply the maximum principle to the above sub-solution to get

$$|\nabla \mathrm{R}|^2 \leq C_1 e^{rt/2}$$

The above estimate can be bootstrapped to get bounds on all higher derivatives,

Proposition 19. *Let $g(t)$ be a solution to the normalised Ricci flow on a closed surface with $r < 0$. Then for each positive integer k $\exists C_k \leq \infty$ such that for all $t \in [0, \infty)$*

$$\sup_{x \in \mathcal{M}} \left|\nabla^k R(x,t)\right|^2 \leq C_k e^{rt/2}.$$

Using these bounds, one may extend the lifetime of existence indefinitely and $g(t)$ converges in every C^k norm to a *bona fide* limit. However we avoid presenting the details of this proof as they are beyond the scope of this document.

Proposition 20 (Long-time Existence)**.** *If $(\mathcal{M}, g_0)$ is a closed Riemannian surface, a unique solution $g(t)$ of the normalised Ricci flow exists for all time and satisfies $g(0) = g_0$.*

As a consequence of exponentially decaying bounds on R and its derivatives, we have proven that $\lim_{t\to\infty} \mathrm{R} = r$ uniformly in all derivatives. This concludes the proof of Theorem 8.

6 The cases of Zero and Positive Average Curvature

We have proven that the Ricci flow uniformises the metric when its average scalar curvature is negative. Topologically, there are infinitely many non-homeomorphic orientable closed surfaces of negative average curvature, classified according to genus. Therefore we are only left with two cases— surfaces of genus 0 and 1, which respectively uniformise to the sphere and the plane. The proof of these two cases of the Uniformisation theorem using the Ricci flow are fairly more complicated than the case we have proven.

The case of $r = 0$ (genus 1) uses similar techniques and estimates as we have used in the case of $r < 0$, but is slightly more involved. The case of $r > 0$ on the other hand is significantly more complicated and require estimates on other geometric quantities that we have not considered here (such as surface entropy and Harnack estimates). Interestingly, although Hamilton showed that the Ricci flow uniformses the metric in this case, the proof involved the Kazdan-Warner identity that is a consequence of the uniformisation theorem. However, Chen, Lu and Tian [5] provided a proof of the theorem independent of the Kazdan-Warner identity.

7 Notation/Appendix

- $T_p\mathcal{M}$ is the tangent space to a manifold $\mathcal{M}$ at the base point p.
- $T\mathcal{M}$ and $T^*\mathcal{M}$ are the tangent and cotangent bundles of the manifold $\mathcal{M}$.
- $C^\infty(\mathcal{V})$ denotes the space of sections of vector bundle $\mathcal{V}$ (this notation is used when the base space is clear). In this notation, $C^\infty(T\mathcal{M})$ is the space of smooth vector fields, $C^\infty(S_2T^*\mathcal{M})$ is the space of symmetric $(2,0)$-tensors and $C^\infty(S_2^+T^*\mathcal{M})$ is the space of positive definite symmetric $(2,0)$-tensors.
- Laplace operator: The well known Laplacian generalises to Riemannian manifolds. When operating on scalar fields, it is often referred to as the *Laplace-Beltrami* operator while it goes by the name of a *connection* or *rough* Laplacian when acting on sections of other vector bundles.

  The Laplace-Beltrami operator on scalar fields may be expressed in coordinates as

  $$\Delta_g f = \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}g^{ij}\partial_j f\right).$$

  Here, $\det g$ is the determinant of the metric represented as a matrix in coordinates.

  Another useful way of thinking about the Laplacian is that it captures the deviation of a function from its local average in the following sense:

  $$\Delta f(x) = \lim_{\epsilon\to 0}\frac{2n}{r^2}\left(\frac{1}{A(S_r)}\int_{S_r} f(y)\,\mathrm{d}y - f(x)\right).$$

- Harmonic coordinate system: This system of coordinates plays a significant role in general relativity and geometric analysis. A coordinate system on a manifold $(\mathcal{M},g)$ is *harmonic* if each coordinate function, $x^i:\mathcal{M}\to\mathbb{R}$ is a harmonic function i.e. $\Delta_g x^i = 0$. In coordinates:

  $$0 = \Delta_g x^i = g^{jk}(\partial_j\partial_k - \Gamma^l_{jk}\partial_l)x^i = -g^{jk}\Gamma^i_{jk}.$$

  The existence of harmonic coordinates follows from existence theory of elliptic PDEs [7, 13].

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