Schoen and Yau's Proof of the Positive Mass Theorem

Gokul Nair

April 2021

1 Introduction

The positive mass theorem is a foundational result in General relativity, which broadly asserts that an 'isolated system', a region of positive curvature, always has a 'non-zero gravitational effect' far away.
It is easy to motivate the theorem using a Newtonian gravitational analogy [1]: consider an integrable matter distribution function, $\rho(x)$ in $\mathbb{R}^3$. Asymptotically, the Newtonian potential, $\varphi(x) = -\frac{m}{r} + O(r^{-2})$, where $m = \int \rho$ is the 'total mass' of the system. Clearly, if $\rho \geq 0$, then $m \geq 0$, i.e. far enough away, an observer would experience an attractive force. In general relativity, by virtue of the Einstein constraint equations, the scalar curvature, R takes the role of ρ. The analogous question, in this case is: "if $R \geq 0$, is the ADM-mass non-negative?"

In 1979, Schoen and Yau proved the positive mass theorem for asymptotically Schwarzschild manifolds [2] and in 1981, they proved the theorem for general asymptotically flat manifolds [3]. Independently, in 1981, Witten proved the theorem for manifolds that admit a spin structure [4].
In this document, we will present the proof in the case of asymptotically Schwarzschild metrics.

2 Asymptotically flat manifolds

Definition 1. *On $\mathbb{R}^3 \setminus \{0\}$ (or $\mathbb{R} \times \mathbb{S}^2$), the spatial Schwarzschild metric is given by*

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij},$$

where δ_{ij} is the Kronecker delta function.

Remark. This metric is related to the spatial part of the standard form of the Schwarzschild metric, $\mathrm{d}s^2 = \left(1 - \frac{2m}{\tilde{r}}\right)^{-1} \mathrm{d}\tilde{r} + \tilde{r}^2 \,\mathrm{d}\Omega^2$, by the coordinate change $\tilde{r} = r\left(1 + \frac{m}{2r}\right)^2$.
We model the effect of an isolated system by requiring that asymptotically, the manifold looks like Schwarzschild space, i.e. as though the system behaves like a spherically symmetric object with some effective mass.

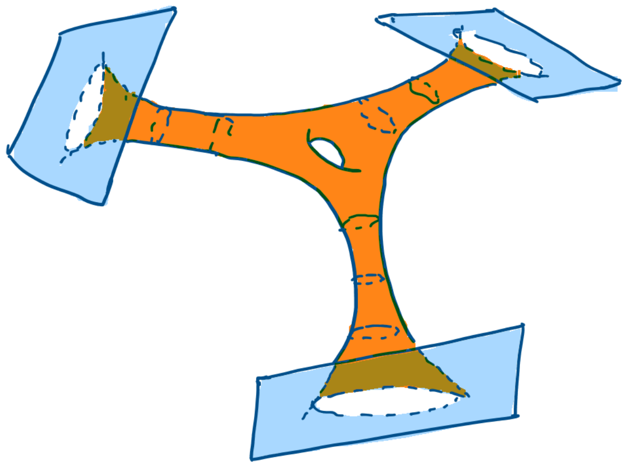


Figure 1: An example of a 2 dimensional asymptotically flat manifold with three ends. Ends are marked in blue whereas the compact set/bulk is marked in orange.

An oriented 3-manifold (without boundary), N is said to be *asymptotically flat* (or more specifically, *asymptotically Schwarzschild*) if there is a compact set $C \subset N$ such that $N \setminus C = \bigcup_{i=1}^{k} N_i$, where each *end*, N_i is diffeomorphic to $\mathbb{R}^3 \setminus B$, where B is a Euclidean ball. Furthermore, in these coordinates, the metric on the end N_k has the form,

$$g_{ij} = \left(1 + \frac{m_k}{2r}\right)^4 \delta_{ij} + p_{ij},$$

where for positive constants k_1, k_2, k_3,

$$|p_{ij}| \leq \frac{k_1}{1+r^2}, \quad |\partial p_{ij}| \leq \frac{k_2}{1+r^3}, \quad \left|\partial^2 p_{ij}\right| \leq \frac{k_1}{1+r^4}.$$

m_k is referred to as the 'total mass' of the end N_k. Here $r = |x|$, in the flat coordinates under the Euclidean norm. From this it is clear that asymptotically $g \simeq \delta + (2m_k/r)\delta + p = O(1)$, $g^{-1} = O(1)$ and $\partial g = O(1/r^2)$, so the connection coefficients, $\Gamma \simeq g^{-1}(\partial g + \partial g - \partial g) = O(1/r^2)$ and the curvature tensors $R \simeq \partial\Gamma = O(1/r^3)$.

We can now state the main theorem,

Theorem 1 (Positive mass). *Let* $\mathrm{d}s^2$ *be an asymptotically flat metric on an oriented 3-manifold* N*. If the scalar curvature,* $R \geq 0$ *on N, then* $m_k \geq 0$ *on each end* N_k*.*

3 Proof of the theorem

The proof consists of three main steps and relies on many 'well known' results in geometry. We always work in a fixed end, N_k with asymptotically flat coordinates $\{x^1, x^2, x^3\}$ on $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$ where $B_{\sigma_0}(0) = \{|x| < \sigma_0\}$ (Euclidean ball). Drop the subscript k on the mass m_k. The overarching idea of the proof is to take $m < 0$ and arrive at a contradiction.

Step 1

We show that if $m < 0$ then there is an asymptotically flat metric $\mathrm{d}\tilde{s}^2$ that is conformal to $\mathrm{d}s^2$ with scalar curvature $\tilde{R} \geq 0$ on N and $\tilde{R} > 0$ outside a compact set of N_k, but still having mass $\tilde{m} < 0$.

Proof. The Laplacian on $\mathbb{R}^3 \setminus B_{\sigma_0}$ is given by

$$\Delta\varphi = \frac{1}{\sqrt{\det g_{ij}}}\partial_i(\sqrt{\det g_{ij}}g^{ij}\partial_j\varphi),$$

where Einstein summation is adopted and g^{ij} denotes the inverse of g_{ij}. To estimate $\det g_{ij}$, we note that

$$\begin{aligned}
\det g_{ij} &= \det \begin{pmatrix} \left(1+\frac{m}{2r}\right)^4 + O(1/r^2) & O(1/r^2) & O(1/r^2) \\ O(1/r^2) & \left(1+\frac{m}{2r}\right)^4 + O(1/r^2) & O(1/r^2) \\ O(1/r^2) & O(1/r^2) & \left(1+\frac{m}{2r}\right)^4 + O(1/r^2) \end{pmatrix} \\
&= \left(1+\frac{m}{2r}\right)^{12} + \frac{3}{r^2}\left(1+\frac{m}{2r}\right)^8 \\
&= (1+\frac{m}{2r})^{12} + O(1/r^2) = 1 + \frac{6m}{r} + O(1/r^2),
\end{aligned}$$

therefore, $\sqrt{\det g_{ij}} = 1 + \frac{3m}{r} + O(1/r^2)$. We also have,

$$g^{ij} = \left(1-\frac{m}{2r}\right)^4 \delta_{ij} + O(1/r^2) = \left(1-\frac{2m}{r}\right)\delta_{ij} + O(1/r^2)$$

plugging all these estimates into the Laplacian of $1/r$,

$$\begin{aligned}
\Delta(1/r) &= \left(1+\frac{3m}{r}+O(1/r^2)\right)^{-1}\partial_i\left[\left(1+\frac{3m}{r}+O(1/r^2)\right)\left(\left(1-\frac{2m}{r}\right)\delta^{ij}+O(1/r^2)\right)\partial_j(1/r)\right] \\
&= \left(1+\frac{3m}{r}+O(1/r^2)\right)^{-1}\partial_i\left[\left(1+\frac{3m}{r}+O(1/r^2)\right)\left(\left(1-\frac{2m}{r}\right)\delta^{ij}+O(1/r^2)\right)\frac{-x^j}{r^3}\right] \\
&= \left(1+\frac{3m}{r}+O(1/r^2)\right)^{-1}\partial_i\left[\left(1+\frac{3m}{r}+O(1/r^2)\right)\left(\left(1-\frac{2m}{r}\right)\frac{-x^i}{r^3}+O(1/r^5)\right)\right] \\
&= \left(1+\frac{3m}{r}+O(1/r^2)\right)^{-1}\partial_i\left[-\frac{x^i}{r^3}-\frac{Mx^i}{4}+O(1/r^5)\right] \\
&= \left(1+\frac{3m}{r}+O(1/r^2)\right)^{-1}\left[\frac{m}{r^4}+O(1/r^6)\right] \\
&= \frac{m}{r^4}+O(1/r^5),
\end{aligned}$$

where we used the fact that $\sum_i \partial_i(x^i/r^3) = 0$ and $\sum_i \partial_i(x^i/r^4) = 1/r^4$.

Now, since $m < 0$, we can for large enough $\sigma > \sigma_0$ say that

$$\Delta(1/r) < 0 \quad \text{for } r \geq \sigma.$$

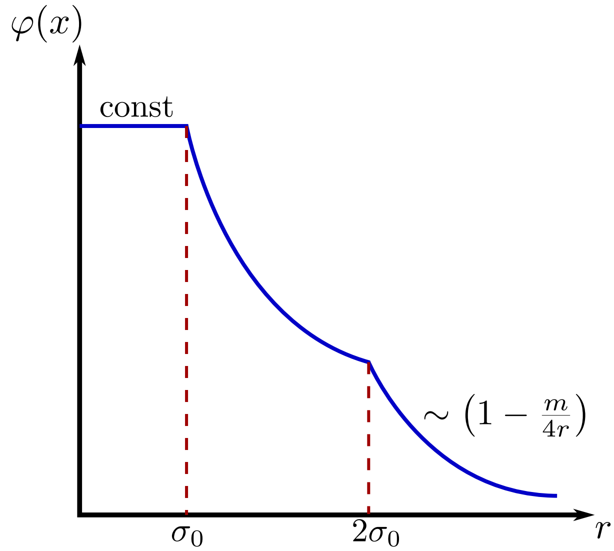


Figure 2: Conformal factor φ: Constant inside B_{σ_0} and other ends, decays like $\left(1-\frac{m}{4r}\right)$ asymptotically. Note that this is only a representative diagram, since the function is actually smooth.

Define $t_0 = -\frac{m}{8\sigma_0}$ and $\zeta(t)$ smooth,

$$\zeta(t) = \begin{cases} t & t < t_0 \\ 3t_0/2 & t > 2t_0 \end{cases}$$

and $\zeta'(t) \geq 0$, $\zeta''(t) \leq 0$ for $t \in (0, \infty)$. Now define the conformal factor $\varphi : N \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} 1 + \frac{3t_0}{2} & x \in N \setminus N_k \\ 1 + \zeta\left(-\frac{m}{4r}\right) & x \in N_k \end{cases}$$

From a direct computation and the fact that $\Delta(1/r) < 0$, we can show that $\Delta\varphi \leq 0$ on N and $\Delta\varphi < 0$ for $r > 2\sigma$. Now define the metric $\mathrm{d}\tilde{s}^2 = \varphi^4 \,\mathrm{d}s^2$. From the curvature formula for conformal metrics,

$$\tilde{R} = \varphi^{-5}[-8\Delta\varphi + R\varphi],$$

we have $\tilde{R} \geq 0$ on N and $\tilde{R} > 0$ on $r > 2\sigma$ (i.e. outside a compact set).

Since φ is constant on every other end N_i $i \neq k$, $\mathrm{d}\tilde{s}^2$ is a constant multiple of $\mathrm{d}s^2$ there. On N_k, for $r > \sigma_0$,

$$\begin{aligned} \tilde{g}_{ij} &= \left(1 - \frac{m}{4r}\right)^4 \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} + O(1/r^2) \\ &= \left(1 + \frac{m}{4r}\right)^4 \delta_{ij} + O(1/r^2). \end{aligned}$$

This is because,

$$\left(1 - \frac{m}{4r}\right)^4 \left(1 + \frac{m}{2r}\right)^4 = 1 + \frac{m}{r} - \frac{m^2}{8r^2} + O(1/r^3)$$

$$\left(1+\frac{m}{4r}\right)^4 = 1+\frac{m}{r}+\frac{3m^2}{8r^2}+O(1/r^3)$$

so $\left(1-\frac{m}{4r}\right)^4\left(1+\frac{m}{2r}\right)^4 = \left(1+\frac{m}{4r}\right)^4 + O(1/r^2)$.

Notice that the metric $\mathrm{d}\tilde{s}^2$ corresponds to an asymptotically flat metric with mass $\tilde{m} = m/2 < 0$. $\square$

We replace $\mathrm{d}s^2$ with the metric $\mathrm{d}\tilde{s}^2$ computed above, but will still refer to it as $\mathrm{d}s^2$ for convenience.

Step 2

For the rest of the proof, we 'extend' the asymptotic coordinates on $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$ into the region corresponding to $B_{\sigma_0}(0)$, so that $\{x^1, x^2, x^3\}$ covers an entire 3-space.

We show that there exists a complete area minimising (with repsect to $\mathrm{d}s^2$) surface S, properly embedded in N so that (i) $S \cap (N \subset N_k)$ is compact and (ii) $S \cap N_k$ lies between two Euclidean planes in 3-space.
In other words, the minimal surface is contained inside a 'horizontal slab' in N_k and does not extend infinitely into any other end. The idea is to construct this surface as the appropriate limit of some sequence of compact minimal surfaces.

Proof. Let $\sigma > 2\sigma_0$ and define C_σ to be Euclidean circle of radius σ lying in the x^1x^2-plane. Solve the Plateau problem to get an area-minimising surface S_σ, with $\partial S_\sigma = C_\sigma$[1].

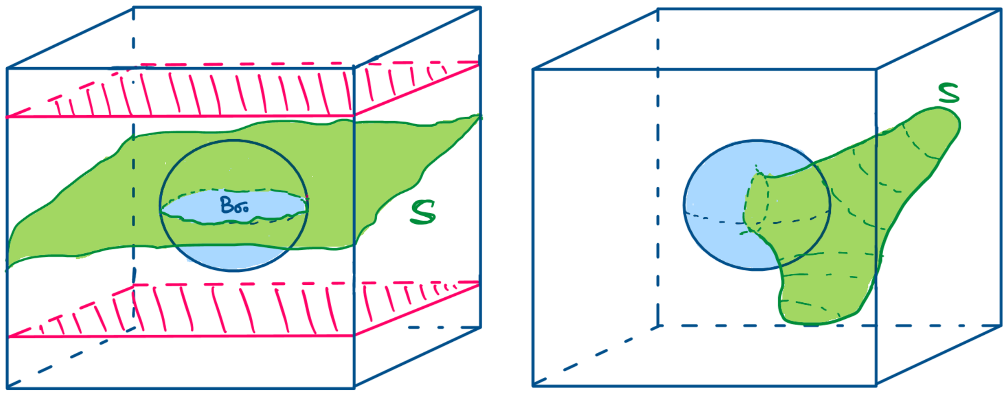
(i) First, we show that there is some compact set $K_0 \subset N$ such that for all $\sigma > 2\sigma_0$, $S_\sigma \cap (N \setminus N_k) \subset K_0$. We will show that due to asymptotic flatness, Euclidean spheres are convex for large enough radii.
Let $N_i \simeq \mathbb{R}^3 \setminus B_{\tau_0}(0)$, $i \neq k$ be another end of N with coordinates y^1, y^2, y^3. Compute the Hessian of $|y|^2$:

$$\begin{aligned}
\nabla^2_{ij} |y|^2 &= \frac{\partial}{\partial y^i}\left(\frac{\partial}{\partial y^j}|y|^2\right) - \nabla_{\frac{\partial}{\partial y^i}}\left(\frac{\partial}{\partial y^j}\right)|y|^2 \qquad (*)\\
&= \frac{\partial^2 |y|^2}{\partial y^i \partial y^j} - \Gamma^k_{ij}\partial_k |y|^2 \\
&= \delta_{ij} - 2\Gamma^k_{ij} y^k \\
&= \delta_{ij} + O(1/|y|),
\end{aligned}$$

where we used the estimate for the connection coefficients from earlier. So there is some $\tau_1 > \tau_0$, such that $|y| \geq \tau_1$, $\nabla^2 |y|^2 > 0$ i.e. $|y|^2$ is convex.

Suppose the sequence $\{S_\sigma \cap N_i\}$ were not uniformly contained in some compact set $K_0 \subset N$ i.e. the sequence 'runs off' to infinity in the end N_i, $i \neq k$. Then, there is some σ_1 such that the surface S_{σ_1} first touches the surface ∂B_{τ_2}, $\tau_2 > \tau_1$. This intersection is in the interior of S_{σ_1} since the boundary, C_{σ_1} lies in N_k and ∂B_{τ_2} is convex from the previous argument. However,

[1] reference Colding Minicozzi



(a) (b)

Figure 3: (a) In our end, N_k, S is contained in a slab (marked in red); (b) The parts of S, lying in other ends , N_i $i \neq k$ are compact.

this contradicts the *convex hull property* [5], which states that the first point of intersection of a convex surface approaching a minimal surface has to be on the boundary of the latter.
Thus, repeating this argument for every other end, $S_\sigma \cap (N \setminus N_k)$ is contained in some compact set K_0.
(ii) Now we focus on $S_\sigma \cap N_k$, in particular, to show that $\exists h > \sigma_0$ such that for all $\sigma > 2\sigma_0$

$$N_k \cap S_\sigma \subset E_h,$$

where the *slab* (or sandwich depending on your taste) $E_h := \{x \in \mathbb{R}^3 : |x^3| \leq h\}$. The idea is to apply the maximum principle to the coordinate function x^3 restricted to $S_\sigma \cap N_k$.

Compute $\nabla^2 x^3$ using the same formula as $(*)$,

$$\nabla^2_{ij} x^3 = 0 - \Gamma^3_{ij} \frac{\partial x^3}{\partial x^3} = -\Gamma^3_{ij},$$

since every other term is zero. Also notice that,

$$g^{ij} = \left(1 - \frac{m}{2r}\right)^4 \delta^{ij} + O(1/r^2) = \left(1 - \frac{2m}{r}\right) \delta^{ij} + O(1/r^2), \quad \frac{\partial g_{ij}}{\partial x^l} = -\frac{2mx^l}{r^3} \delta_{ij} + O(1/r^3).$$

Explicitly calculating Γ^3_{ij},

$$\begin{aligned}
\Gamma^3_{ij} &= \frac{1}{2} g^{3m} \left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right) \\
&= \frac{1}{2} \left[\left(1 - \frac{2m}{r}\right) \delta^{3m} + O(1/r^2) \right] \left[-\frac{2m}{r^3} (x^j \delta_{im} + x^i \delta_{jm} - x^m \delta_{ij}) + O(1/r^3) \right] \\
&= -\frac{m}{r^3} (x^j \delta_{i3} + x^i \delta_{j3} - x^3 \delta_{ij}) + O(1/r^3).
\end{aligned}$$

So we have $\nabla^2_{ij} x^3 = \frac{m}{r^3} (x^j \delta_{i3} + x^i \delta_{j3} - x^3 \delta_{ij}) + O(1/r^3)$.

Now consider the function x^3 on $S_\sigma \cap N_k$. Since this is a compact set, x^3 attains a maximum,

$$\overline{h} := \max_{S_\sigma \cap N_k} \{x^3\}.$$

We want to show that $\overline{h}$ is uniformly bounded (independent of σ). If $\overline{h} \leq \sigma_0$, that uniform bound could simply be σ_0, so we look at the case $\overline{h} > \sigma_0$.

Suppose the maximum is attained at $x_0 \in S_\sigma \cap N_k$. Since $\overline{h} > \sigma_0$, x_0 is away from the boundary curve C_σ (which lies on the x^1x^2-plane) and the other boundary of $S_\sigma \cap N_k$, which lies on ∂B_{σ_0}. In other words, x_0 is a local maximum, and so x^3 has to have 'zero slope' there (and parallel to the x^1x^2-plane). So, the tangent space $T_{x_0}S_\sigma$ is spanned by $\left.\frac{\partial}{\partial x^1}\right|_{x_0}, \left.\frac{\partial}{\partial x^2}\right|_{x_0}$. Locally extend these to vector fields defined on a neighbourhood in S_σ and denote them by v_1, v_2. Let q_{ij} be the metric induced by $\mathrm{d}s^2$ on S_σ in the v_1, v_2 coordinates. Denote the induced connection on S_σ by $\tilde{\nabla}$:

$$\begin{aligned}
\nabla_{v_i} v_j &= (\nabla_{v_i} v_j)^\top + (\nabla_{v_i} v_j)^\perp \\
&= \tilde{\nabla}_{v_i} v_j + \langle \nabla_{v_i} v_j, \nu \rangle \, \nu \\
&= \tilde{\nabla}_{v_i} v_j + h_{ij}\nu,
\end{aligned}$$

where ν is the unit normal field of $\S_\sigma$ and h_{ij} is its second fundamental form. Therefore,

$$\begin{aligned}
\tilde{\nabla}_{ij} x^3 &= v_i(v_j(x^3)) - (\tilde{\nabla}_{v_i} v_j) x^3 \\
&= v_i(v_j(x^3)) - (\nabla_{v_i} v_j) x^3 + h_{ij}\nu(x^3) \\
&= \nabla^2_{ij} x^3 + h_{ij}\nu(x^3)
\end{aligned}$$

Tracing the above with respect to q^{ij}, and using the fact that S_σ is a minimal surface, i.e. $\mathrm{tr}_q(h_{ij}) = q^{ij}h_{ij} = 0$ yields

$$q^{ij}\tilde{\nabla}^2_{ij} x^3 = q^{ij}\nabla^2_{ij} x^3 + q^{ij}h_{ij}\nu(x^3) = q^{ij}\nabla^2_{ij} x^3.$$

Since $T_{x_0}S_\sigma$ is just a flat slice of $\mathbb{R}^3$, $q^{ij} = \delta^{ij}$, $i = 1, 2$ and $j = 1, 2$. So,

$$\left. q^{ij}\tilde{\nabla}^2_{ij} x^3 \right|_{x_0} = \delta^{ij}\frac{m}{r^3}(x_0^j \delta_{i3} + x_0^i \delta_{j3} - \overline{h}\delta_{ij}) + O(1/r^3),$$

where the summation for i, j is from 1 to 2. This gives

$$\left. q^{ij}\tilde{\nabla}^2_{ij} x^3 \right|_{x_0} = -\frac{2m\overline{h}}{r^3} + O(1/r^3).$$

Since $m < 0$, for sufficiently large $\overline{h}$, we get $\left. q^{ij}\tilde{\nabla}^2_{ij} x^3 \right|_{x_0} > 0$. However, since x_0 is a maximum, $\left. \tilde{\nabla}^2_{ij} x^3 \right|_{x_0}$ must be negative semi-definite and $\left. q^{ij}\tilde{\nabla}^2_{ij} x^3 \right|_{x_0} \leq 0$ (since q^{ij} is a positive definite symmetric matrix). This is a contradiction, so $\overline{h}$ cannot be arbitrarily large.
Replacing maximum with minimum in this argument yields a lower bound for x^3.

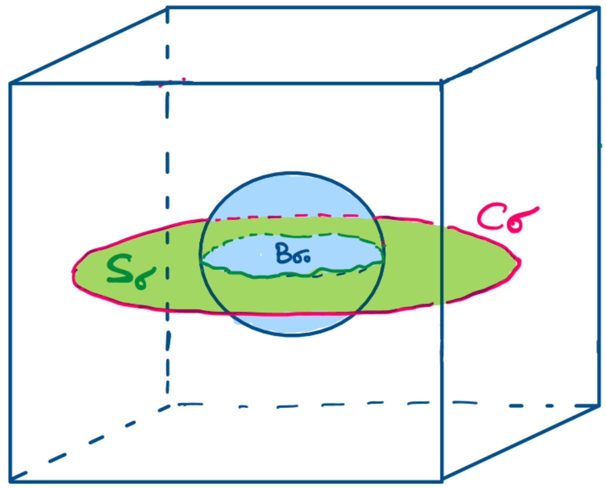


Figure 4: Constructing S as a sequence of solutions to Plateau's problem for the circle C_σ.

Now, we extract a subsequence $\{\sigma_i\}$ such that $S_{\sigma_i} \to S$ that is properly embedded and satisfies the desired properties. This is an involved step that requires ideas from the regularity theory for minimal surfaces [5, 2, 6]. The general idea is to represent the surfaces S_σ as graphs of functions and use the results derived above to get bounds on the derivatives of these functions. This, along with the regularity estimate can be used to extract the required subsequence such that $S_{\sigma_i} \to S$ in C^2 on compact subsets of N. $\square$

Step 3

The final step is to show that the minimal surface S cannot exist. This is done by using the second variation inequality and a clever application of the Gauss-Bonnet theorem, to arrive at a contradiction thus finishing the proof of the theorem.

Define,

$$S_{(\sigma)} := [S \cap (N \setminus N_k)] \cup [S \cap B_\sigma(0)].$$

$\{S_{(\sigma)}\}$ forms an exhaustion of S.

Claim 2. *Let $\sigma \geq \sigma_0$. There exists a constant C_2 independent of σ such that,*

$$\text{Area}(S_{(\sigma)}) \leq C_2\sigma^2$$

Proof. If S intersects $\partial B_\sigma(0)$ transversally, the intersection is a union of C^2 simple closed curves[2]. These curves bound a domain $\Omega \subset \partial B_\sigma(0)$ that shares the same boundary as $S_{(\sigma)}$. Using the area-minimising property of S,

$$\text{Area}(S_{(\sigma)}) \leq \text{Area}(\Omega) \leq \text{Area}(\partial B_\sigma(0)).$$

Since for $\sigma > \sigma_0$, the metric is uniformly, Euclidean this implies the required inequality for transverse intersections. Since non-transverse intersections can be 'deformed' by arbitrarily

[2]figure

small amounts to become transverse, we can use an approximation argument to extend the inequality to all intersections. □

Lemma 3. *For $a > 2$,*

$$\int_S \frac{\mathrm{d}r}{1+r^a} \leq C_2\sigma_0^2 + C_2 a \int_{\sigma_0}^{\infty} \frac{t^{1+a}}{(1+t^a)^2}\,\mathrm{d}t$$

Proof.

$$\begin{aligned}
\int_S \frac{\mathrm{d}r}{1+r^a} &= \int_{S_{(\sigma_0)}} \frac{\mathrm{d}r}{1+r^a} + \int_{\sigma_0}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_{(t)}} \frac{\mathrm{d}r}{1+r^a}\right)\mathrm{d}t && \text{(FTC)}\\
&\leq \mathrm{Area}(S_{(\sigma_0)}) + \int_{\sigma_0}^{\infty} \frac{1}{1+t^a}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Area}(S_{(t)})\right)\mathrm{d}t && \text{(see Remark)}\\
&\leq C_2\sigma_0^2 + a\int_{\sigma_0}^{\infty} \frac{t^{a-1}}{(1+t^a)^2}\mathrm{Area}(S_{(t)})\,\mathrm{d}t && \text{(Integrate by parts)}\\
&\leq C_2\sigma_0^2 + C_2 a\int_{\sigma_0}^{\infty} \frac{t^{a+1}}{(1+t^a)^2}\,\mathrm{d}t.
\end{aligned}$$

□

Remark. In the second line of the above proof, we use an inequality that can be obtained from this calculuation:

$$\begin{aligned}
\int_{S_{(t+h)}} \frac{\mathrm{d}r}{1+r^a} - \int_{S_{(t)}} \frac{\mathrm{d}r}{1+r^a} &= \int_{S_{(t+h)}\setminus S_{(t)}} \frac{\mathrm{d}r}{1+r^a} && \text{(since } S_{(t)} \subseteq S_{(t+h)})\\
&\leq \sup_{r\in S_{(t+h)}\setminus S_{(t)}} \left\{\frac{1}{1+r^a}\right\} \int_{S_{(t+h)}\setminus S_{(t)}} 1\,\mathrm{d}r\\
&\leq \frac{1}{1+t^a}\mathrm{Area}(S_{(t+h)} \setminus S_{(t)}).
\end{aligned}$$

Dividing by $h > 0$ and taking $h \to 0$, yields the inequality used in the second line of the proof.

Lemma 4. *For $\sigma_2 > \sigma_1 > \sigma_0$,*

$$\int_{S_{(\sigma_2)}\setminus S_{(\sigma_1)}} \frac{\mathrm{d}r}{r^2} \leq 2C_2\log(\sigma_2/\sigma_1).$$

Proof.

$$\begin{aligned}
\int_{S_{(\sigma_2)}\setminus S_{(\sigma_1)}} \frac{\mathrm{d}r}{r^2} &= \int_{\sigma_1}^{\sigma_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{S_{(t)}} \frac{\mathrm{d}r}{r^2}\right)\mathrm{d}t\\
&\leq \int_{\sigma_1}^{\sigma_2} \frac{1}{t^2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Area}(S_{(t)}))\,\mathrm{d}t && \text{(same as Remark)}\\
&= \int_{\sigma_1}^{\sigma_2} \frac{2}{t^3}\mathrm{Area}(S_{(t)})\,\mathrm{d}t && \text{(Integrate by parts)}\\
&\leq \int_{\sigma_1}^{\sigma_2} \frac{2}{t^3}C_2 t^2\,\mathrm{d}t = 2\log(\sigma_2/\sigma_1).
\end{aligned}$$

□

The Second Variation inequality

By studying the stability of the minimal surface S constructed in the previous step, we will see that it has to have positive total curvature. Using the Gauss-Bonnet theorem cleverly, we will show that this is not possible, and that will give us the required contradiction.

Let $\{e_1, e_2, e_3\}$ be an orthonormal frame field defined locally for N. We denote the sectional curvature of the plane spanned by $\{e_i, e_j\}$ as

$$K_{ij} = \sec\{e_i, e_j\}.$$

Sectional curvature is related to the Riemann curvature, Rm in the following way,

$$K_{ij} = \langle \mathrm{Rm}(e_i, e_j)e_j, e_j \rangle ,$$

where the inner product is with respect to the metric, $\mathrm{d}s^2$ on N. Clearly, $K_{ii} = 0$.
The (0,2)-Ricci curvature tensor Rc is related to Rm in the following way,

$$\mathrm{Rc}(e_i, e_k) = \sum_{j=1}^{3} \langle \mathrm{Rm}(e_i, e_j)e_j, e_k \rangle .$$

We can relate sectional curvature to Rc in the following way,

$$\mathrm{Rc}(e_i, e_i) = \sum_{j=1}^{3} \langle \mathrm{Rm}(e_i, e_j)e_j, e_i \rangle = \sum_{j=1}^{3} K_{ij}.$$

Finally, we write the scalar curvature, R.

$$\mathrm{R} = \sum_{i=1}^{3} \mathrm{Rc}(e_i, e_i) = \sum_{i=1}^{3}\sum_{j=1}^{3} K_{ij} = 2(K_{12} + K_{23} + K_{31}),$$

since $K_{ii} = 0$ and K_{ij} is symmetric.

Remark. In Schoen and Yau's paper, they use the notation $\mathrm{Rc}(e_i)$ to mean $\mathrm{Rc}(e_i, e_i)$. Also, they do not include the factor of 2 in the formula relating R to the sectional curvatures. This is a minor difference that doesn't affect the rest of the proof.

Let ν be the unit normal vector field to S and consider the frame $e_1, e_2, e_3 = \nu$ restricted to S (e_1 and e_2 are tangent to S). The second fundamental form of S, denoted by A has the following matrix components in the basis $\{e_1, e_2\}$,

$$h_{ij} = \langle \nabla_{e_i}\nu, e_j \rangle$$

and $|A|^2 = \sum_{i,j=1}^{2} h_{ij}^2$. Since S is minimal, $\mathrm{tr}(A) = h_{11} + h_{22} = 0$.

The second variation inequality (or stability condition) for a minimal surface is given by[3]

$$\int_S f[\Delta f + (\mathrm{Rc}(\nu, \nu) + |A|^2)f] \leq 0,$$

[3]refer CM

where $f \in C_c^2(S)$ (twice continuously differentiable functions with compact support on S). Using approximations, we can take f to be Lipschitz with compact support.
Integrating the above by parts yields,

$$\int_S (\mathrm{Rc}(\nu,\nu) + |A|^2) f^2 \leq \int_S |\Delta f|^2 \tag{$**$}$$

Gauss's equation says that

$$\langle \mathrm{Rm}(e_1,e_2)e_2, e_1\rangle = \left\langle \tilde{\mathrm{Rm}}(e_1,e_2)e_2, e_1\right\rangle + h_{12}^2 - h_{11}h_{22},$$

where $\tilde{\mathrm{Rm}}$ is the Riemann curvature of S. Notice that the term of the left hand side is the sectional curvature K_{12} and the first term on the right side is the sectional curvature of S or the Gaussian curvature, K (since e_1 and e_2 are tangent). Therefore,

$$\begin{aligned} K &= K_{12} + h_{11}h_{22} - h_{12}^2 \\ &= K_{12} - h_{11}^2 - h_{12}^2 && (S \text{ is minimal, } h_{11} = -h_{22}) \\ &= K_{12} - \frac{1}{2}|A|^2. && (\text{Using symmetry of } A \text{ and minimality of } S) \end{aligned}$$

Hence $\frac{1}{2}|A|^2 = K_{12} - K$. Substituting this in the second variation inequality,

$$\int_S \left(\mathrm{Rc}(\nu,\nu) + K_{12} - K + \frac{1}{2}|A|^2\right) f^2 \leq \int_S |\nabla f|^2 \tag{1}$$

Since $K_{ii} = 0$, we have

$$\mathrm{Rc}(\nu,\nu) = \sum_{j=1}^{3} K_{3j} = K_{13} + K_{23}.$$

Using this formula in the above inequality yields,

$$\int_S \left(\frac{1}{2}R - K + \frac{1}{2}|A|^2\right) f^2 \leq \int_S |\nabla f|^2 \tag{$***$}$$

Now, we make clever choices of f to get some useful estimates. For $\sigma > \sigma_0$, define the cut-off function

$$\varphi = \begin{cases} 1 & \text{on } S_{(\sigma)} \\ \frac{\log \frac{\sigma^2}{r}}{\log \sigma} & \text{on } S_{(\sigma^2)} \setminus S_{(\sigma)} \\ 0 & \text{on } S \setminus S_{(\sigma^2)} \end{cases}$$

Let $g \leq 1$ be Lipschitz on S and have the property that $|g| = 1$ outside a compact subset of S.
Define $f = \varphi g$ (it is Lipschitz since φ and g are).
Using this form of f in $(**)$ gives

$$\begin{aligned} \int_S (\mathrm{Rc}(\nu,\nu) + |A|^2)\varphi^2 g^2 &\leq \int_S |\nabla(\varphi g)|^2 \\ &= \int_S |g\nabla\varphi|^2 + 2\int_S (g\nabla\varphi)\cdot(\varphi\nabla g) + \int_S |\varphi\nabla g|^2 \\ &\leq 2\int_S g^2|\nabla\varphi|^2 + 2\int_S \varphi^2|\nabla g|^2 && (\text{Cauchy's inequality}) \end{aligned}$$

$\nabla\varphi = -\frac{1}{\log\sigma}\frac{\nabla r}{r}$ on $S_{(\sigma^2)} \setminus S_{(\sigma)}$ and zero elsewhere. Also, $\nabla r = x/r$, and computing the norm in $\mathrm{d}s^2$:

$$\begin{aligned} |\nabla r|^2 &= \frac{1}{r^2} g_{ij} x^i x^j \\ &= \left(1 + \frac{m}{2r}\right)^4 + \frac{1}{r^2} h_{ij} x^i x^j \\ &\leq 1 + \frac{2m}{r} + \frac{k_1}{1+r^2}, \end{aligned}$$

which is bounded on $r > \sigma_0$ so we have $|\nabla r|^2 < C_3$. Rearranging the earlier inequality,

$$\int_{S_{(\sigma)}} |A|^2 g^2 \leq \frac{2C_3}{(\log\sigma)^2} \int_{S_{(\sigma^2)} \setminus S_{(\sigma)}} \frac{1}{r^2} + 2\int_S |\nabla g|^2 + \int_S |\mathrm{Rc}(\nu,\nu)|\, g^2.$$

Using Lemma 4, with $\sigma_2 = \sigma^2$ and $\sigma_1 = \sigma$, we get

$$\int_{S_{(\sigma)}} |A|^2 g^2 \leq \frac{2C_2C_3}{\log\sigma} + 2\int_S |\nabla g|^2 + \int_S |\mathrm{Rc}(\nu,\nu)|\, g^2.$$

Since this is true for all $\sigma > \sigma_0$, we can take $\sigma \to \infty$ (using monotone convergence)

$$\int_S |A|^2 g^2 \leq 2\int_S |\nabla g|^2 + \int_S |\mathrm{Rc}(\nu,\nu)|\, g^2.$$

Set $g = 1$ and since $\mathrm{Rc}(\nu,\nu) = O(1/r^3)$, the second term converges (Lemma 3) so we have

$$\int_S |A|^2 < \infty.$$

We have,

$$|K| \leq |K_{12}| + \left|h_{11}h_{22} - h_{12}^2\right| \leq |K_{12}| + \frac{1}{2}\left|h_{11}^2 - 2h_{12}^2 + h_{22}^2\right| \leq |K_{12}| + |A|^2.$$

Since $|K_{12}| = O(1/r^3)$ (bound on Riemann curvature) from Lemma 3 again, $\int_S |K_{12}| < \infty$, therefore we get the bound,

$$\int_S |K| < \infty. \tag{$\dagger$}$$

Now, in $(***)$ take $f = \varphi$

$$\int_{S_{(\sigma)}} \left(R - K + \frac{1}{2}|A|^2\right) + \int_{S_{(\sigma^2)} \setminus S_{(\sigma)}} \left(R - K + \frac{1}{2}|A|^2\right) \frac{\log(\sigma^2/r)}{\log\sigma} \leq \int_{S_{(\sigma^2)} \setminus S_{(\sigma)}} \frac{1}{(\log\sigma)^2} \frac{|\nabla r|^2}{r^2}.$$

Taking $\sigma \to \infty$,

$$\int_S R - K + \frac{1}{2}|A|^2 \leq 0.$$

From assumption, $R \geq 0$ and $R > 0$ outside a compact subset of S, therefore,

$$\int_S K > 0. \tag{$\dagger\dagger$}$$

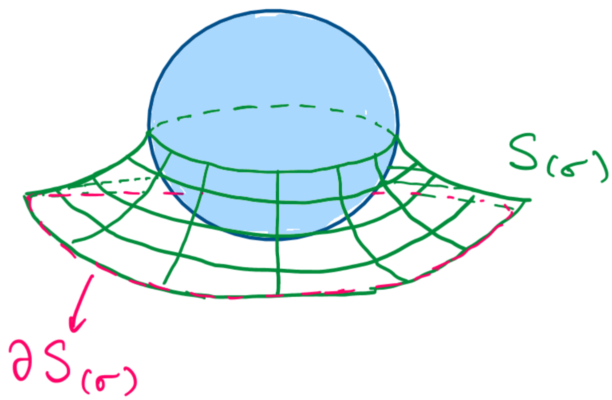


Figure 5: At large radii, $S_{(\sigma)}$ approaches a disk.

Arriving at a Contradiction

We state the Cohn-Vossen inequality, which is analogous to the Gauss-Bonnet theorem in the case of non-compact surfaces.

Theorem 5 (Cohn-Vossen inequality)**.** *For a complete 2-surface, S with finite total curvature and Euler Characteristic, $\chi(S)$,*

$$\int_S K \leq 2\pi\chi(S).$$

From (††), $\chi(S) > 0$ therefore $\chi(S) \geq 1$ (since the Euler characteristic is an integer). For a surface $\chi(S) = 1 - \operatorname{rank}(H_1)$ and since $\operatorname{rank}(H_1) \geq 0$, we get $\chi(S) = 1$ and $H_1(S) = 0$. This implies that S is homeomorphic to $\mathbb{R}^2$.[4]

Claim 6.

$$\int_S K \leq 0.$$

Schoen and Yau provide two proofs of this claim. The first proof is much shorter and more succinct than the second, but it relies on certain esoteric results from other papers, whereas the second proof mostly relies on the Gauss-Bonnet theorem, so we choose that route. Schoen and Yau's original presentation of this proof, however, is very tedious so we take the approach in [5].

Sketch of Proof. The main idea is to apply the Gauss-Bonnet theorem to $S \cap B_\sigma(0)$. A key observation is that using the fact that S is homeomorphic to $\mathbb{R}^2$, it can be shown that for large r, S is a graph of some function u over the x^1x^2-plane. Using the estimates on u from the last paragraph of Step 2, we can show that the boundary $\partial(S \cap B_\sigma(0))$, has geodesic curvature $k_g = \sigma^{-1} + O(1/\sigma)$ in S. Computing the integral around the boundary,

$$\int_{\partial(S \cap B_\sigma(0))} k_g = (2\pi\sigma + O(1))(\sigma^{-1} + O(1/\sigma)) = 2\pi + O(1/\sigma).$$

[4]reference classification theorem/Uniformisation theorem/stackexchange

Applying the Gauss-Bonnet theorem to $S \cap B_\sigma(0)$,

$$\int_{S\cap B_\sigma(0)} K = 2\pi - \int_{\partial(S\cap B_\sigma(0))} k_g = O(1/\sigma).$$

Taking the limit $\sigma \to \infty$ yields the result. $\square$

4 Conclusions

A closely related result, known as *positive mass rigidity*, which addresses the case when the ADM-mass is zero, is also proven in [2], but we will not present that here.

Schoen and Yau's proof of the positive mass theorem is a fascinating application of the theory of minimal surfaces. This powerful technique however does not easily generalise to dimensions greater than 8. More recently, however, there has been development in this direction [1, 7].

References

[1] Dan A Lee. *Geometric relativity*, volume 201. American Mathematical Soc., 2019.

[2] Richard Schoen and Shing-Tung Yau. On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*, 65(1):45–76, 1979.

[3] Richard Schoen and Shing-Tung Yau. Proof of the positive mass theorem. ii. *Communications in Mathematical Physics*, 79(2):231–260, 1981.

[4] Edward Witten. A new proof of the positive energy theorem. *Communications in Mathematical Physics*, 80(3):381–402, 1981.

[5] Tobias H Colding and William P Minicozzi. *A course in minimal surfaces*, volume 121. American Mathematical Soc., 2011.

[6] Richard Schoen, Leon Simon, and Shing-Tung Yau. Curvature estimates for minimal hypersurfaces. *Acta Mathematica*, 134(1):275–288, 1975.

[7] Richard Schoen and Shing-Tung Yau. Positive scalar curvature and minimal hypersurface singularities. *arXiv preprint arXiv:1704.05490*, 2017.