§5.2 Name: Solutions

Fall MATH 1120 Lec003 22-26 August 2022

Constructing Accurate Graphs of Antiderivatives

Review

- **Fundamental Theorem of Calculus:** If f is a continuous function on $[a, b]$ and F is an antiderivative of f, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

- **Total Change Theorem:** If f is continuously differentiable on $[a, b]$ with derivative f', then

$$f(b) - f(a) = \int_a^b f'(x)\,\mathrm{d}x.$$

Main Concepts

- Properties of the antiderivative F over some interval $[a, b]$ can be deduced from the signed area under the graph of f.
- A function can have multiple antiderivatives. However, to uniquely determine an antiderivative, you just need to know its value at a single point. In fact, if F and G are antiderivatives of the same function f, then $G - H$ is a constant.
- For a continuous function f, you can define its *integral function* A as

$$A(x) = \int_a^x f(t)\,\mathrm{d}t.$$

$A(x)$ can be thought of as the "Total signed area under the graph of f starting from a up to x".

Activities

Activity 5.1.2

Suppose that the function $y = f(x)$ is given by the graph shown in 1, and that the pieces of f are either portions of lines or portions of circles. In addition, let F be an antiderivative of f and say that $F(0) = -1$. Finally, assume that for $x \leq 0$ and $x \geq 7$, $f(x) = 0$.

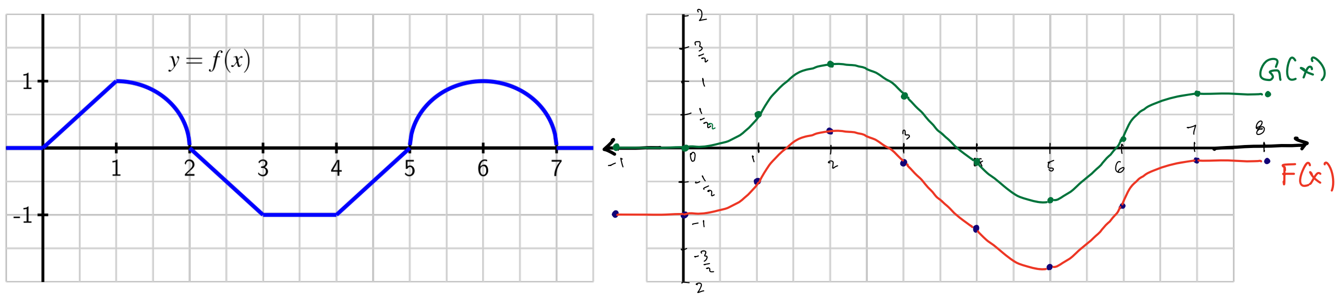


Figure 1: The graph of $y = f(x)$.

(a) On what interval(s) is F an increasing function? On what intervals is F decreasing?

F is increasing when f is positive and decreasing when f is negative. So, F increases on (0,2) & (5,7) and decreases on (2,5)

(b) On what interval(s) is F concave up? concave down? neither?

Concave up/down can be deduced from whether f is increasing or decreasing. Thus, F is concave up for (0,1) & (4,6). It is concave down for (1,3) & (6,7)

(c) At what point(s) does F have a relative minimum? a relative maximum?

F will have a relative max/min at points where f is zero. If f is increasing @ that point, it's a min & vice versa. So x = 2 is a max and x = 5 is a min

(d) Use the given information to determine the exact value of $F(x)$ for $x = 1, 2, \ldots, 7$. In addition, what are the values of $F(-1)$ and $F(8)$?

** Remember $F(0) = -1$ ** The rest is just adding up areas of triangles, rectangles & circles.

$F(-1) = -1$ $F(1) = -\frac{1}{2}$ $F(2) = \frac{\pi}{4} - \frac{1}{2}$ $F(3) = \frac{\pi}{4} - 1$ $F(4) = \frac{\pi}{4} - 2$

$F(5) = \frac{\pi}{4} - \frac{5}{2}$ $F(6) = \frac{\pi}{2} - \frac{5}{2}$ $F(7) = \frac{3\pi}{4} - \frac{5}{2}$ $F(8) = \frac{3\pi}{4} - \frac{5}{2}$

(e) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = F(x)$ on the axes provided, being sure to indicate the behavior of F for $x < 0$ and $x > 7$. Clearly indicate the scale on the vertical and horizontal axes of your graph.

Look at Pg 2

(f) What happens if we change one key piece of information: in particular, say that G is an antiderivative of f and $G(0) = 0$. How (if at all) would your answers to the preceding questions change? Sketch a graph of G on the same axes as the graph of F you constructed in (e).

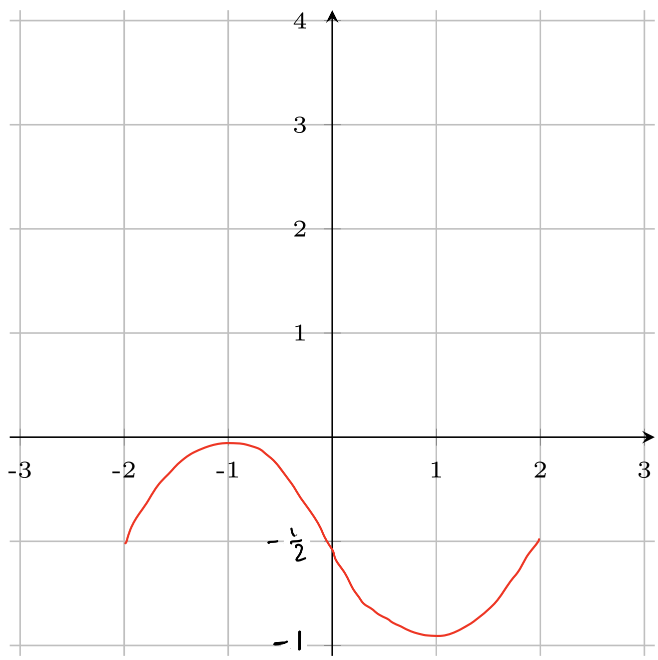
(a) (b) (c) remain unchanged. The graph gets shifted upwards by 1.

$G(x) = F(x) + 1$

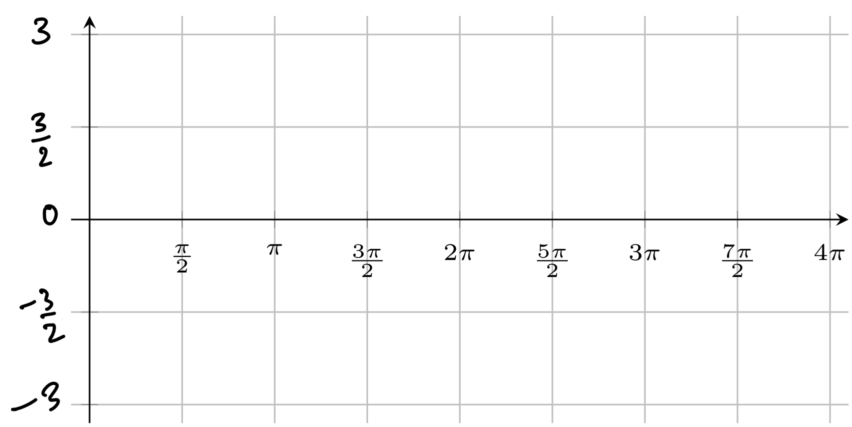
Activity 5.1.3

For each of the following functions, sketch an accurate graph of the antiderivative that satisfies the given initial condition. In addition, sketch the graph of two additional antiderivatives of the given function, and state the corresponding initial conditions that each of them satisfy. If possible, find an algebraic formula for the antiderivative that satisfies the initial condition.

(a) original function: $g(x) = |x| - 1$; initial condition: $G(-1) = 0$; interval for sketch: $[-2, 2]$

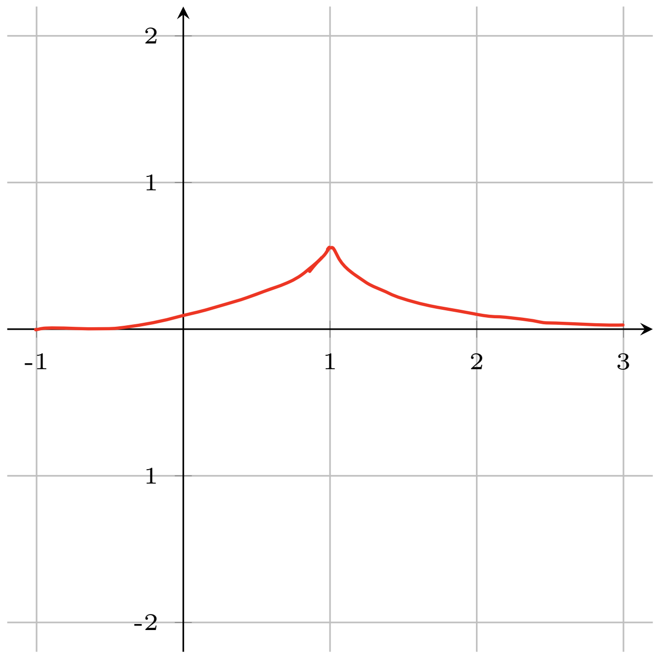


(b) original function: $h(x) = \sin(x)$; initial condition: $H(0) = 1$; interval for sketch: $[0, 4\pi]$



(c) original function: $p(x) = \begin{cases} x^2, & \text{if } 0 < x < 1 \\ -(x-2)^2, & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$; initial condition: $P(0) = 1$;

interval for sketch: $[-1, 3]$



Activity 5.1.4

Suppose that g is given by the graph at left in 2 and that A is the corresponding integral function defined by $A(x) = \int_1^x g(t)\,dt$.

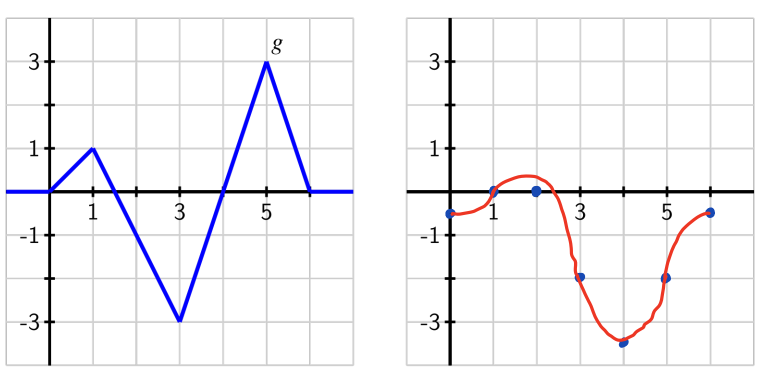


Figure 2: At left, the graph of $y = g(t)$; at right, axes for plotting $y = A(x)$, where A is defined by the formula $A(x) = \int_1^x g(t)\,dt$.

(a) On what interval(s) is A an increasing function? On what intervals is A decreasing? Why?

Incr : $(0, 1.5)$, $(4, 6)$ Decr : $(1.5, 4)$

(b) On what interval(s) do you think A is concave up? concave down? Why?

Up : (0,1) , (3,5)

(c) At what point(s) does A have a relative minimum? a relative maximum?

$x = 1.5 \to$ max

$x = 4 \to$ min

(d) Use the given information to determine the exact values of $A(0)$, $A(1)$, $A(2)$, $A(3)$, $A(4)$, $A(5)$, and $A(6)$.

$A(0) = -0.5$, $A(1) = -\frac{1}{2}$, $A(2) = 0$, $A(3) = 0$, $A(4) = -2$, $A(5) = -3.5$, $A(6) = -2$

(e) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = A(x)$ on the axes provided, being sure to indicate the behavior of A for $x < 0$ and $x > 6$.

Page 5

(f) How does the graph of B compare to A if B is instead defined by $B(x) = \int_0^x g(t)\,dt$.

Its shifted up by $\frac{1}{2}$. $B(x) = A(x) + \frac{1}{2}$