§5.2 Name: SOLUTIONS

Fall MATH 1120 Lec003 22-26 August 2022

The Second Fundamental Theorem of Calculus

Main Concepts

- **Second Fundamental Theorem of Calculus:** If f is a continuous function and c is any constant, then f has a unique antiderivative A that satisfies $A(c) = 0$ given by

$$A(x) = \int_c^x f(t)\,\mathrm{d}t.$$

- **Differentiating an Integral Function** Integration and Differentiation can be viewed as "inverse processes" of each other:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_c^x f(t)\,\mathrm{d}t\right] = f(x)$$

Activities

Activity 5.2.2

Suppose that f is the function given in 1 and that f is a piecewise function whose parts are either portions of lines or portions of circles, as pictured. In addition, let A be the function

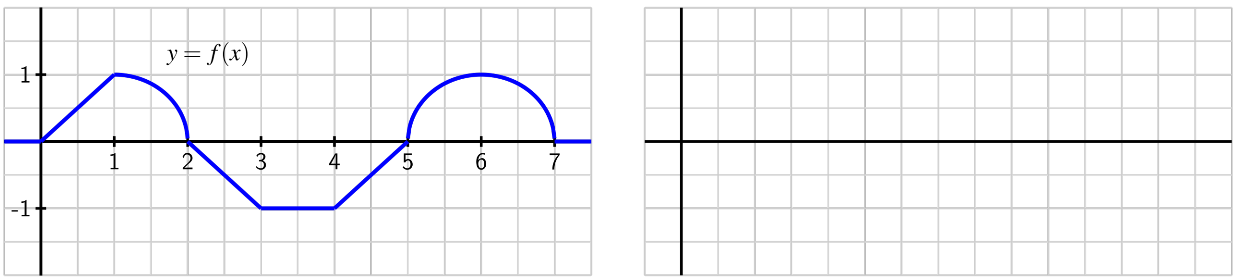

Figure 1: At left, the graph of $y = f(x)$. At right, axes for sketching $y = A(x)$.

defined by the rule $A(x) = \int_2^x f(t)\,dt$.

(a) What does the Second FTC tell us about the relationship between A and f?

$A'(x) = f(x)$

(b) Compute $A(1)$ and $A(3)$ exactly.

$A(1) = \int_2^1 f(t)\,dt = -\int_1^2 f(t)\,dt = -\pi/4$ $\quad A(3) = \int_2^3 f(t)\,dt = -\frac{1}{2}$

(c) Sketch a precise graph of $y = A(x)$ on the axes at right that accurately reflects where A is increasing and decreasing, where A is concave up and concave down, and the exact values of A at $x = 0, 1, \ldots, 7$.

$A(0) = -\pi/4 - \frac{1}{2}$ $\quad A(1) = -\frac{\pi}{4}$ $\quad A(2) = 0$ $\quad A(3) = -\frac{1}{2}$ $\quad A(4) = -\frac{3}{2}$

$A(5) = -2$ $\quad A(6) = -2 + \pi/4$ $\quad A(7) = -2 + \pi/2$

(d) How is A similar to, but different from the function F that you found in Activity 5.1.2?

$F - A = \frac{\pi}{4} - \frac{1}{2}$

(e) With as little additional work as possible, sketch the precise graphs of the functions $B(x) = \int_3^x f(t)\,dt$ and $C(x) = \int_1^x f(t)\,dt$. Justify your results with at least one sentence of explanation.

They differ by constants. $\quad B - A = \frac{1}{2}$

$C - A = \pi/4$

Activity 5.2.3

Suppose that $f(t) = \frac{t}{1+t^2}$ and $F(x) = \int_0^x f(t)\,dt$.

(a) On the axes at left in 2, plot a graph of $f(t) = \frac{t}{1+t^2}$ on the interval $-10 \le t \le 10$. Clearly label the vertical axes with appropriate scale.

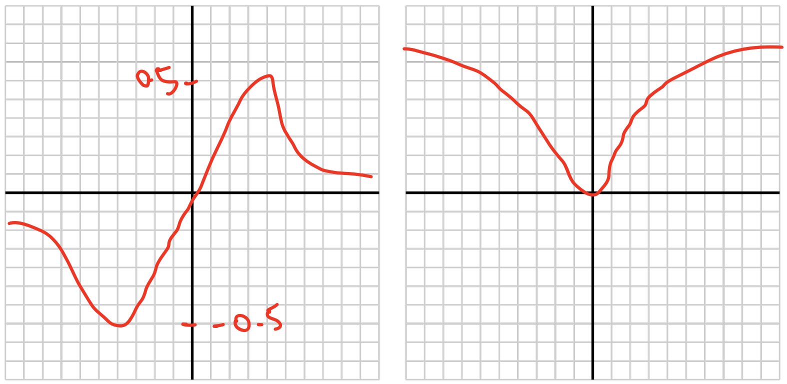

Figure 2: Axes for plotting f and F.

(b) What is the key relationship between F and f, according to the second FTC?

$F' = f$

(c) Use the first derivative test to determine the intervals on which F is increasing and decreasing.

F is increasing when $f > 0$ $(x > 0)$ & decreasing when $f < 0$ $(x < 0)$

(d) Use the second derivative test to determine the intervals on which F is concave up and concave down. Note that $f'(t)$ can be simplified to be written in the form $f'(t) = \frac{1-t^2}{(1+t^2)^2}$.

F is CCU when $f' > 0$ i.e. $-1 < x < 1$

F is CCD when $f' < 0$ i.e. $x > 1$ & $x < -1$

(e) Using technology appropriately, estimate the values of $F(5)$ and $F(10)$ through appropriate Riemann sums.

(f) Sketch an accurate graph of $y = F(x)$ on the right hand axes provided, and clearly label the vertical axes with appropriate scale.

Activity 5.2.4

Evaluate each of the following derivatives and definite integrals. Clearly cite whether you use the First or Second FTC in so doing.

(a) $\frac{d}{dx}\left[\int_4^x e^{t^2}\,dt\right]$

Apply 2^{nd} FTC :

$= e^{x^2}$

(b) $\int_{-2}^x \left[\frac{d}{dt}\frac{t^4}{1+t^4}\right]dt$

Apply 1^{st} FTC

$= \frac{x^4}{1+x^4} - \frac{16}{17}$

(c) $\frac{d}{dx}\left[\int_x^1 \cos(t^3)\,dt\right]$ $= -\frac{d}{dx}\left[\int_1^x \cos t^3\,dt\right]$ Apply 2^{nd} FTC

$= -\cos x^3$

(d) $\int_3^x \left[\frac{d}{dt}\ln(1+t^2)\,dt\right]$

Apply 1st FTC

$= \ln(1+x^2) - \ln 10$

(e) $\frac{d}{dx}\left[\int_4^{x^2} \sin(t^2)\,dt\right]$ $\left(u = x^2 \qquad \frac{d}{dx} = \frac{du}{dx}\frac{d}{du} = 2x\frac{d}{du} \right)$

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then $2x\frac{d}{du}\int_4^u \sin t^2\,dt = 2x \sin x^4$

Apply 2^{nd} FTC