§5.3 **Name:** ____________________

Fall MATH 1120 Lec003 29 August - 2 September 2022

Integration by Substitution

Review

- **Chain rule:** Suppose u is differentiable at x and f is differentiable at $u(x)$. Then the derivative of the composite function $f(u(x))$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(u(x))\right] = f'(u(x))u'(x)$$

- $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x \quad \frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x \quad \frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x \quad \frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}} \quad \frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$

Main Concepts

- **Integrating a function composed with a linear function:** Let $h(x) = f(ax+b)$ and let F denote the (algebraic) antiderivative of f, then the (general) antiderivative of h is given by:

$$H(x) = \frac{1}{a}F(ax+b) + C.$$

- **Integration by Substitution:** Suppose g is a differentiable function with continuous derivative, g' and f is a continuous function. Then,

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(u)\,\mathrm{d}u = f(u) + C = f(g(x)) + C,$$

with the substitution $u = g(x)$.

Activities

Activity 5.3.2

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

(a) $\int \sin(8-3x)\,\mathrm{d}x$

Solution: $\frac{1}{3}\cos(8-3x)+C$

(b) $\int \sec^2(4x)\,\mathrm{d}x$

Solution: $\frac{1}{4}\tan(4x)+C$

(c) $\int \frac{1}{11x-9}\,dx$

Solution: $\frac{1}{11}\ln|11x-9|+C$

(d) $\int \csc(2x+1)\cot(2x+1)\,\mathrm{d}x$

Solution: $-\frac{1}{2}\csc(2x+1)+C$

(e) $\int \frac{1}{\sqrt{1-16x^2}}\,\mathrm{d}x$

Solution: $\frac{1}{4}\arcsin 4x+C$

(f) $\int 5^{-x}\,\mathrm{d}x$

Solution: -$\frac{1}{\ln 5}5^{-x}+C$

Activity 5.3.3

Evaluate each of the following indefinite integrals by using these steps:

- Find two functions within the integrand that form (up to a possible missing constant) a function-derivative pair;
- Make a substitution and convert the integral to one involving u and du;
- Evaluate the new integral in u;
- Convert the resulting function of u back to a function of x by using your earlier subsitution;
- Check your work by differentiating the function of x. You should come up with the integrand originally given.

(a) $\int \frac{x^2}{5x^3+1}\,\mathrm{d}x$

Solution: Let $u=5x^3+1$, then $\mathrm{d}u=15x^2\,\mathrm{d}x$. Using this, we have,

$$\int \frac{x^2}{5x^3+1}\,\mathrm{d}x=\frac{1}{15}\int \frac{15x^2}{5x^3+1}\,\mathrm{d}x=\frac{1}{15}\int \frac{1}{u}\,\mathrm{d}u=\frac{1}{15}\ln|u|+C=\frac{1}{15}\ln\left|5x^3+1\right|+C.$$

(b) $\int e^x \sin(e^x)\,\mathrm{d}x$

Solution: Let $u = e^x$, consequently $\mathrm{d}u = e^x\,\mathrm{d}x$. Then,

$$\int e^x \sin e^x\,\mathrm{d}x = \int \sin u\,\mathrm{d}u = -\cos u + C = -\cos(e^x) + C.$$

(c) $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x$

Solution: Let $u = \sqrt{x}$, then $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$.

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x = 2\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\,\mathrm{d}x = 2\int \cos u\,\mathrm{d}u = 2\sin u + C = 2\sin\sqrt{x} + C.$$

Activity 5.3.4

Evaluate each of the following definite integrals exactly through an appropriate u-substitution.

(a) $\int_1^2 \frac{x}{1+4x^2}\,\mathrm{d}x$

Solution: $\frac{1}{8}(\ln(17) - \ln(5))$. Use $u = 1 + 4x^2$

(b) $\int_0^1 e^{-x}(2e^{-x}+3)^9\,\mathrm{d}x$

Solution: $-\frac{1}{20}(2\frac{1}{e}+3)^{10} + \frac{1}{20}5^{10}$. Use $u = 2e^{-x}+3$.

(c) $\int_{2/\pi}^{4/\pi} \frac{\cos\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x$

Solution: $1 - \frac{\sqrt{2}}{2}$. Use $u = 1/x$.