§5.4 Name: SOLUTIONS

Fall MATH 1120 Lec003 29 August - 2 September 2022

# Integration by Parts

## Review

- **Product Rule:** Let $u$ and $v$ be differentiable functions. Then the derivative of the product is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}(u \cdot v) = u \cdot \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}x} \cdot v$$

## Main Concepts

- **Integrating by parts** can be used to compute integrals of the form $\int f(x)g'(x)\,\mathrm{d}x$:

$$\int f(x)\,g'(x)\,\mathrm{d}x = f(x)\,g(x) - \int g(x)\,f'(x)\,\mathrm{d}x$$

It is also expressed as

$$\int u\,\mathrm{d}v = u\,v - \int v\,\mathrm{d}u$$

where $u = f(x)$, $\mathrm{d}u = f'(x)\,\mathrm{d}x$, $v = g(x)$ and $\mathrm{d}v = g'(x)\,\mathrm{d}x$.

## Activities

### Activity 5.4.2

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

(a) $\int te^{-t}\,\mathrm{d}t$

$u = t \qquad dv = e^{-t}dt$

$du = dt \qquad v = -e^{-t}$

$\int te^{-t}dt = -te^{-t} - \int(-e^{-t})\,dt$

$= -te^{-t} - e^{-t} + C$

(b) $\int 4x\sin(3x)\,\mathrm{d}x$

$u = x \qquad dv = \sin 3x\,dx$

$du = dx \qquad v = -\frac{1}{3}\cos 3x$

$4\int x\sin 3x = 4\left[x\left(-\frac{1}{3}\cos 3x\right) - \int\left(-\frac{1}{3}\cos 3x\right)dx\right]$

$= -\frac{4}{3}x\cos 3x + \frac{4}{9}\sin 3x + C$

(c) $\int z\sec^2(z)\,\mathrm{d}z$

$u = z \qquad dv = \sec^2 z\,dz$

$du = dz \qquad v = \tan z$

$\int z\sec^2 z\,dz = z\tan z - \int \tan z\,dz$

$= z\tan z - \int \frac{\sin z}{\cos z}\,dz$ let $u = \cos z$, $du = -\sin z\,dz$

$-\int \frac{\sin z}{\cos z}dz = \int \frac{du}{u} = \ln|u| = \ln|\cos z|$

therefore

$\int z\sec^2 z\,dz = z\tan z + \ln|\cos z| + C$

(d) $\int x\ln(x)\,\mathrm{d}x$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{x^2}{2}$

$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

Activity 5.4.2

Evaluate each of the following indefinite integrals, using the provided hints.

(a) Evaluate $\int \arctan(x)\,dx$ by using Integration by Parts with the substitution $u = \arctan(x)$ and $dv = 1\,dx$.

$u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = x$

$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$

let $1+x^2 = u$

$du = 2x\,dx$

$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln|1+x^2|$

so

$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln|1+x^2| + C$

(b) Evaluate $\int \ln(z)\,dz$. Consider a similar substitution to the one in (a).

$u = \ln z \qquad dv = dz$

$du = \frac{1}{z}\,dz \qquad v = z$

$\int \ln z\,dz = z\ln z - \int z\,\frac{1}{z}\,dz$

$= z\ln z - \int dz = z\ln z - z + C$

(c) Use the substitution $z = t^2$ to transform the integral $\int t^3 \sin(t^2)\,dt$ to a new integral in the variable $z$, and evaluate the new integral by parts.

$\int t^3 \sin(t^2)\,dt = \int t \cdot t^2 \sin(t^2)\,dt$

$z = t^2 \qquad dz = 2t\,dt \qquad \frac{1}{2}\,dz = t\,dt$

$\int \underbrace{t \cdot \overbrace{t^2}^{z} \sin(\overbrace{t^2}^{z})\,dt}_{\frac{1}{2}dz} = \frac{1}{2}\int z\sin z\,dz$

let $u = z \qquad dv = \sin z\,dz$

$du = dz \qquad v = -\cos z$

then

$= \frac{1}{2}\left[-z\cos z - \int(-\cos z)\,dz\right]$

$= \frac{1}{2}\left[-z\cos z + \sin z\right]$

$= \frac{1}{2}\left[-t^2\cos t^2 + \sin t^2\right] + C$

(d) Evaluate $\int s^5 e^{s^3}\, ds$ using an approach similar to that described in (c).

$\int s^5 e^{s^3} ds = \int s^2 s^3 e^{s^3} ds \qquad t = s^3 \quad dt = 3s^2 ds \qquad s^2 ds = \frac{dt}{3}$

$\frac{1}{3}\int te^t dt = \frac{1}{3}\left[te^t - e^t\right] + C = \frac{1}{3}\left[s^3 e^{s^3} - e^{s^3}\right] + C$

integrate by parts

(e) Evaluate $\int e^{2t}\cos(e^t)\, dt$. You will find it helpful to note that $e^{2t} = e^t \cdot e^t$.

Same idea as above

$\int e^{2t}\cos(e^t)dt = \int e^t e^t \cos(e^t)\, dt \qquad z = e^t \qquad dz = e^t dt$

$= \int z\cos z\, dz$ then integrate by parts

Activity 5.4.2

Evaluate each of the following indefinite integrals.

(a) $\int x^2 \sin(x)\, dx$ $\quad u = x^2 \quad dv = \sin x\, dx$

$du = 2x\, dx \qquad v = -\cos x \qquad \int x^2 \sin x\, dx = -x^2\cos x + 2\int x\cos x\, dx$

Integrate 2nd term by parts

$u = x \qquad dv = \cos x\, dx$

$du = dx \qquad v = \sin x$

$\int x\cos x\, dx = x\sin x - \int \sin x\, dx$

$= x\sin x + \cos x$

so

$\int x^2 \sin x\, dx = -x^2\cos x + 2x\sin x + \cos x + C$

(b) $\int t^3 \ln(t)\, dt$

$u = \ln t \qquad du = \frac{dt}{t}$

$dv = t^3 dt \qquad v = \frac{t^4}{4}$

$\int t^3 \ln t\, dt = \frac{t^4}{4}\ln t - \int \frac{t^4}{4}\frac{dt}{t}$

$= \frac{t^4}{4}\ln t - \frac{1}{4}\int t^3 dt$

$= \frac{t^4}{4}\ln t - \frac{1}{16}t^4 + C$

(c) $\int e^z \sin(z)\,\mathrm{d}z$ Let's call $I = \int e^z \sin z \, dz$

$u = e^z \quad dv = \sin z \, dz$

$du = e^z dz \quad v = -\cos z$

$I = -e^z \cos z + \int e^z \cos z \, dz$

Now let $u = e^z$ $dv = \cos z dz$ and integrate by parts again

$du = e^z dz \quad v = \sin z$

$I = -e^z \cos z + \left[ e^z \sin z - \int e^z \sin z \, dz \right]$

$I = -e^z \cos z + e^z \sin z - I$

$2I = -e^z \cos z + e^z \sin z$

$I = \frac{1}{2}\left(-e^z \cos z + e^z \sin z\right) + C$

(d) $\int s^2 e^{3s}\,\mathrm{d}s$

$u = s^2 \quad du = 2s \, ds$

$dv = e^{3s} ds \quad v = \frac{1}{3} e^{3s}$

$\int s^2 e^{3s} ds = \frac{1}{3} s^2 e^{3s} - \frac{2}{3} \int s e^{3s} ds$

$u = s \quad du = ds$

$dv = e^{3s} ds \quad v = \frac{1}{3} e^{3s}$

$= \frac{1}{3} s^2 e^{3s} - \frac{2}{3} \left[ \frac{s e^{3s}}{3} - \frac{1}{3} \int e^{3s} ds \right]$

$= \frac{1}{3} s^2 e^{3s} - \frac{2}{9} s e^{3s} + \frac{2}{27} e^{3s} + C$

(e) $\int t \arctan(t)\,\mathrm{d}t$ (**Hint:** At a certain point in this problem, it is very helpful to note that $\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}$.)

$u = \arctan t \quad du = \frac{dt}{1+t^2}$

$dv = t \, dt \quad v = \frac{t^2}{2}$

$\int t \arctan t \, dt = \frac{t^2}{2} \arctan t - \frac{1}{2} \int \frac{t^2}{1+t^2} dt$

$= \frac{t^2}{2} \arctan t - \frac{1}{2} \int 1 - \frac{1}{1+t^2} dt$

$= \frac{t^2}{2} \arctan(t) - \frac{1}{2} t + \frac{1}{2} \arctan(t) + C$