§5.5 **Name:** ______________________

Fall MATH 1120 Lec003 5 September - 9 September 2022

Alternative Methods for Finding Antiderivatives

Main Concepts

- **Partial Fractions:** Used for integrating rational functions, i.e. $\frac{P(x)}{Q(x)}$ where P and Q are polynomials and Q has a higher degree than P. For the types of problems we are interested in, examples/guidelines can be followed to obtain a *partial fraction decomposition*:

  1. Factor the denominator polynomial Q.
  2. Write your fraction as a sum of simpler fractions multiplied with constant coefficients, where the denominators are the factors of $Q(x)$. For example,
  $$\frac{2x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$$
  3. If the denominator $Q(x)$ has quadratic factors, then use $Ax + B$ as one of the coefficients instead of a constant, for example,
  $$\frac{1}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$
  4. If the denominator has the same factor repeated n times, then include that factor n times in the decomposition, but with powers ranging from 1 to n. For example:
  $$\frac{x^2+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}.$$
  5. If the denominator has a mix of the above three scenarios, then combine the guidelines from above appropriately.
  6. Solve for the constants A, B, C etc.

- **Integral Tables and Computer Algebra:** You can use integral tables or computer algebra to solve integrals

Practice Problems

Find the partial fraction decompositions for the following:

1. $\frac{3x+11}{x^2-x-6}$

$$\frac{-1}{x+2} + \frac{4}{x-3}$$

2. $\frac{1}{(x-1)^2(x+1)}$

$$\frac{-1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)}$$

3. $\frac{1}{(x-1)(x^2+1)}$

$$\frac{1}{2(x-1)} - \frac{x+1}{2(x^2+1)}$$

Activities

Activity 5.5.2

For each of the following problems, evaluate the integral by using the partial fraction decomposition provided.

(a) $\int \frac{1}{x^2-2x-3}\,dx$, given that $\frac{1}{x^2-2x-3} = \frac{1/4}{x-3} - \frac{1/4}{x+1}$.

$$\int \frac{1}{x^2-2x-3}\,dx = \frac{1}{4}\int \frac{dx}{x-3} - \frac{1}{4}\int \frac{dx}{x+1}$$

$$= \frac{1}{4}\ln(|x-3|) - \frac{1}{4}\ln(|x+1|) + C$$

$$= \frac{1}{4}\ln\left(\frac{|x-3|}{|x+1|}\right) + C$$

(b) $\int \frac{x^2+1}{x^3-x^2}\,dx$, given that $\frac{x^2+1}{x^3-x^2} = -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$.

$$\int \frac{x^2+1}{x^3-x^2}\,dx = -\int \frac{dx}{x} - \int \frac{dx}{x^3} + \int \frac{2dx}{x-1} = -\ln|x| + \frac{1}{2x^2} + \ln|x-1| + C$$

(c) $\int \frac{x-2}{x^4+x^2}\,\mathrm{d}x$, given that $\frac{x-2}{x^4+x^2} = \frac{1}{x} - \frac{2}{x^2} + \frac{-x+2}{1+x^2}$.

$\int \frac{x-2}{x^4+x^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{2\,dx}{x^2} - \int \frac{x-2}{1+x^2}\,dx$

$= \ln|x| + \frac{2}{x} - \int \frac{x}{1+x^2}\,dx + \int \frac{2}{1+x^2}\,dx \to 2\tan^{-1}x$

$1+x^2 = t \quad dt = 2x\,dx$

$\frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| = \frac{1}{2}\ln|1+x^2|$

$= \ln|x| + \frac{2}{x} - \frac{1}{2}\ln|1+x^2| + 2\arctan x + C$

Activity 5.5.3

For each of the following integrals, evaluate the integral using u-substitution, and/or using an entry from the following integral table:

$$\text{(i)} \int \frac{\mathrm{d}u}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$$

$$\text{(ii)} \int \sqrt{u^2 \pm a^2}\,\mathrm{d}u = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln|u + \sqrt{u^2 \pm a^2}| + C$$

$$\text{(iii)} \int \frac{\mathrm{d}u}{\sqrt{u^2 \pm a^2}} = \ln|u + \sqrt{u^2 \pm a^2}| + C$$

(a) $\int \sqrt{x^2+4}\,\mathrm{d}x$

Use (ii) with "+" and $a = 2$ $u = x$

$\int \sqrt{x^2+4}\,dx = \int \sqrt{u^2+2^2}\,du = \frac{u}{2}\sqrt{u^2+4} + \frac{4}{2}\ln|u+\sqrt{u^2+4}| + C$

$= \frac{x}{2}\sqrt{x^2+4} + 2\ln|x+\sqrt{x^2+4}| + C$

(b) $\int \frac{x}{\sqrt{x^2+4}}\,\mathrm{d}x$

let $x^2+4 = u \quad du = 2x\,dx$

$= \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\frac{\sqrt{u}}{1/2} = \sqrt{x^2+4} + C$

(c) $\int \frac{2}{\sqrt{16+25x^2}}\,\mathrm{d}x$

use (iii) with "+", $a=4$, $u=5x$ $(du=5dx)$

$$\int \frac{2}{\sqrt{4^2+(5x)^2}}\,dx = \frac{2}{5}\int \frac{du}{\sqrt{4^2+u^2}} = \frac{2}{5}\ln\left|u+\sqrt{u^2+4^2}\right| + C$$

$$= \frac{2}{5}\ln\left|5x+\sqrt{25x^2+16}\right| + C$$

(d) $\int \frac{1}{x^2\sqrt{49-36x^2}}\,\mathrm{d}x$

Use (i) with "-", $a=7$ $u=6x$ $du=6dx$

multiply and divide by 36

$$\int \frac{36}{36x^2\sqrt{7^2-(6x)^2}}\,dx = \int \frac{36\,dx}{(6x)^2\sqrt{7^2-(6x)^2}} = \overset{6}{\cancel{36}}\int \frac{1}{u^2\sqrt{7^2-u^2}}\,\frac{dx}{\cancel{6}}$$

$$= 6\int \frac{dx}{u^2\sqrt{7^2-u^2}}$$

$$= -6\,\frac{\sqrt{7^2-u^2}}{7^2 u} + C$$

$$= -\cancel{6}\,\frac{\sqrt{49-36x^2}}{49(\cancel{6}x)}$$

$$= -\frac{\sqrt{49-36x^2}}{49x} + C$$