Alternative Methods for Finding Antiderivatives

Main Concepts

- **Partial Fractions:** Used for integrating rational functions, i.e. $\frac{P(x)}{Q(x)}$ where P and Q are polynomials and Q has a higher degree than P. For the types of problems we are interested in, examples/guidelines can be followed to obtain a *partial fraction decomposition*:

  1. Factor the denominator polynomial Q.
  2. Write your fraction as a sum of simpler fractions multiplied with constant coefficients, where the denominators are the factors of $Q(x)$. For example,
  $$\frac{2x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$$
  3. If the denominator $Q(x)$ has quadratic factors, then use $Ax + B$ as one of the coefficients instead of a constant, for example,
  $$\frac{1}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$
  4. If the denominator has the same factor repeated n times, then include that factor n times in the decomposition, but with powers ranging from 1 to n. For example:
  $$\frac{x^2+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}.$$
  5. If the denominator has a mix of the above three scenarios, then combine the guidelines from above appropriately.
  6. Solve for the constants A, B, C etc.

- **Integral Tables and Computer Algebra:** You can use integral tables or computer algebra to solve integrals

Practice Problems

Find the partial fraction decompositions for the following:

1. $\frac{3x+11}{x^2-x-6}$

2. $\frac{1}{(x-1)^2(x+1)}$

3. $\frac{1}{(x-1)(x^2+1)}$

Activities

Activity 5.5.2

For each of the following problems, evaluate the integral by using the partial fraction decomposition provided.

(a) $\int \frac{1}{x^2-2x-3}\, \mathrm{d}x$, given that $\frac{1}{x^2-2x-3} = \frac{1/4}{x-3} - \frac{1/4}{x+1}$.

(b) $\int \frac{x^2+1}{x^3-x^2}\, \mathrm{d}x$, given that $\frac{x^2+1}{x^3-x^2} = -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$.

(c) $\int \frac{x-2}{x^4+x^2}\,\mathrm{d}x$, given that $\frac{x-2}{x^4+x^2} = \frac{1}{x} - \frac{2}{x^2} + \frac{-x+2}{1+x^2}$.

Activity 5.5.3

For each of the following integrals, evaluate the integral using u-substitution, and/or using an entry from the following integral table:

$$\int \frac{\mathrm{d}u}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$$

$$\int \sqrt{u^2 \pm a^2}\,\mathrm{d}u = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln|u+\sqrt{u^2 \pm a^2}| + C$$

$$\int \frac{\mathrm{d}u}{\sqrt{u^2 \pm a^2}} = \ln|u+\sqrt{u^2 \pm a^2}| + C$$

(a) $\int \sqrt{x^2+4}\,\mathrm{d}x$

(b) $\int \frac{x}{\sqrt{x^2+4}}\,\mathrm{d}x$

(c) $\int \frac{2}{\sqrt{16+25x^2}}\,\mathrm{d}x$

(d) $\int \frac{1}{x^2\sqrt{49-36x^2}}\,\mathrm{d}x$