§6.1 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 12 September - 16 September 2022

Area Between Curves and Arc Length

Main Concepts

- Given two curves $y = f(x)$ and $y = g(x)$ that intersect at two points, you can find the are between the two curves by following the guidelines below:

  1. Draw the curves to visualize the region of interest.
  2. Find the points of intersection of the two curves, denote them by solving $f(x) = g(x)$. Denote the points by $(a, f(a))$ and $(b, f(b))$.
  3. Suppose $f(x) \geq g(x)$ on $a \leq x \leq b$. The area of the region bounded by the curves is given by the integral

$$A = \int_a^b f(x) - g(x)\,\mathrm{d}x.$$

- The same idea can be used to find the area of regions bounded by curves of the form $x = f(y)$ and $x = g(y)$. If the points of intersection are $(f(c), c)$ and $(f(d), d)$, and $f(y) \geq g(y)$ for $c \leq y \leq d$, then the area of the bounded region is given by

$$A = \int_c^d f(y) - g(y)\,\mathrm{d}y$$

- If f is differentiable function on $[a, b]$, the arc-length of the curve $y = f(x)$ is given by the integral

$$\int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x.$$

Activities

Activity 6.1.2

In each of the following problems, our goal is to determine the area of the region described. For each region,

(i) determine the intersection points of the curves,

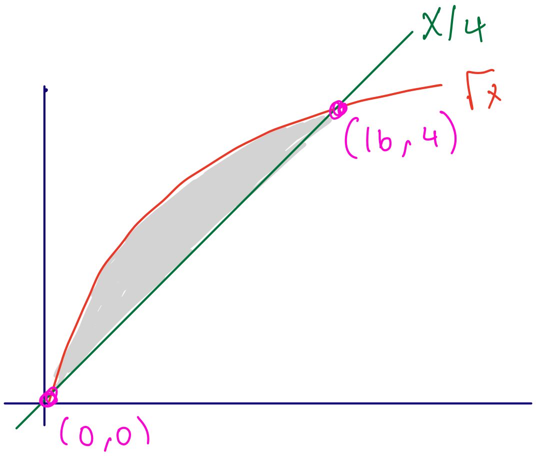
(ii) sketch the region whose area is being found,

(iii) draw and label a representative slice, and

(iv) state the area of the representative slice.

Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area.

(a) The finite region bounded by $y = \sqrt{x}$ and $y = \frac{x}{4}$.



Solving: $\frac{x}{4} = \sqrt{x}$

$\frac{x^2}{16} = x$ $\quad (x^2 - 16x) = 0$

$x(x-16) = 0$

$x = 0$ or $x = 16$

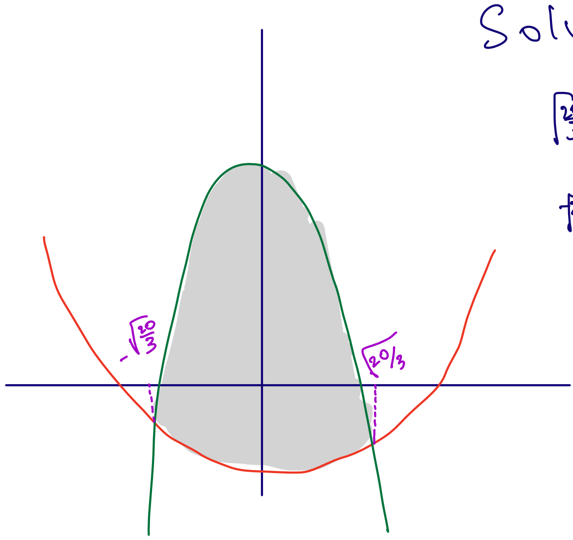
$\int_0^{16} \sqrt{x} - \frac{x}{4}\, dx = 32/3$

(b) The finite region bounded by $y = 12 - 2x^2$ and $y = x^2 - 8$.

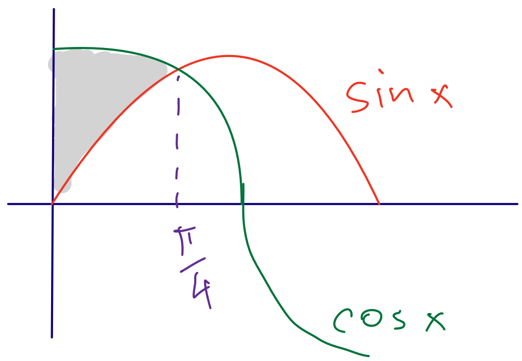
Solving: $12 - 2x^2 = x^2 - 8$

$3x^2 = 20$ $\quad x = \pm\sqrt{\frac{20}{3}}$

$\int_{-\sqrt{\frac{20}{3}}}^{\sqrt{\frac{20}{3}}} 12 - 2x^2 - (x^2 - 8)\, dx = \frac{80\sqrt{5}}{3\sqrt{3}}$



(c) The area bounded bz the y-axis, $y = \cos(x)$ and $y = \sin(x)$, where we consider the region formed by the first positive value of x for which the two curves intersect.

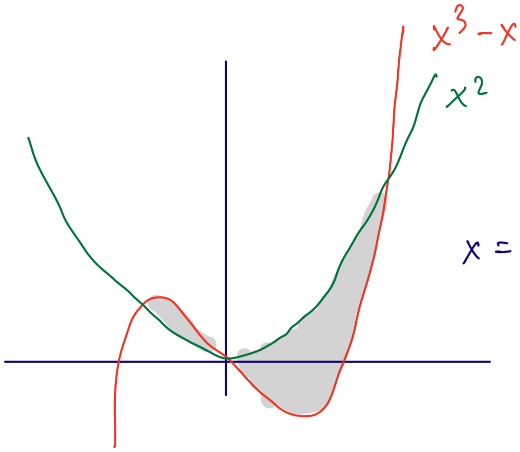


Solving $\sin x = \cos x$

$\Rightarrow x = \pi/4$

$$\int_0^{\pi/4} \cos x - \sin x = \sqrt{2} - 1$$

(d) The finite regions between the curves $y = x^3 - x$ and $y = x^2$.



Solving: $x^3 - x = x^2$

$x(x^2 - x - 1) = 0 \quad x = 0$

$$x = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Total area: $\displaystyle\int_{\frac{1-\sqrt{5}}{2}}^{0} x^3 - x - x^2 \, dx + \int_0^{\frac{1+\sqrt{5}}{2}} x^2 - x^3 + x \, dx$

$= 26/24$

Activity 6.1.3

In each of the following problems, our goal is to determine the area of the region described. For each region,

(i) determine the intersection points of the curves,

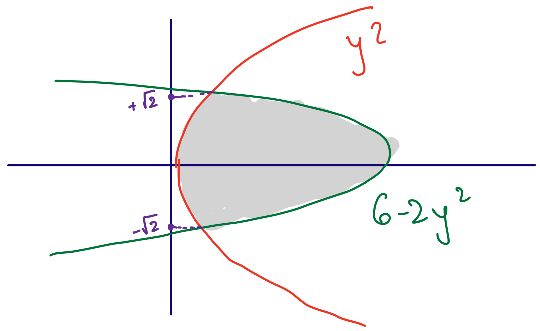
(ii) sketch the region whose area is being found,

(iii) draw and label a representative slice, and

(iv) state the area of the representative slice.

Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area. Note well: At the step where you draw a representative slice, you need to make a choice about whether to slice vertically or horizontally.

(a) The finite region bounded by $x = y^2$ and $x = 6 - 2y^2$.



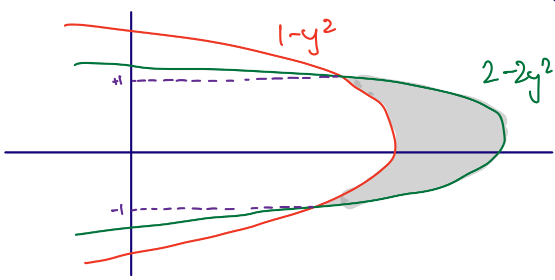
Solving: $y^2 = 6 - 2y^2$ $\quad 3y^2 = 6$

$y = \pm\sqrt{2}$

Area: $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} (6-2y^2) - y^2\,dy$

$= 8\sqrt{2}$

(b) The finite region bounded by $x = 1 - y^2$ and $x = 2 - 2y^2$.



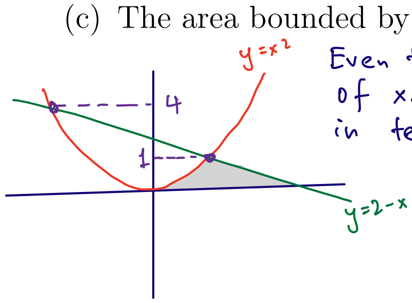
Solving: $1 - y^2 = 2 - 2y^2$

$y^2 = 1 \quad y = \pm 1$

Area: $\displaystyle\int_{-1}^{1} 2 - 2y^2 - (1-y^2)$

$= 4/3$

(c) The area bounded by the x-axis, $y = x^2$ and $y = 2 - x$.



Even though the functions are given in terms of x, this area is easier to compute in terms of a y-integral.

$y = x^2 \longrightarrow x = \sqrt{y}$ (positive x)

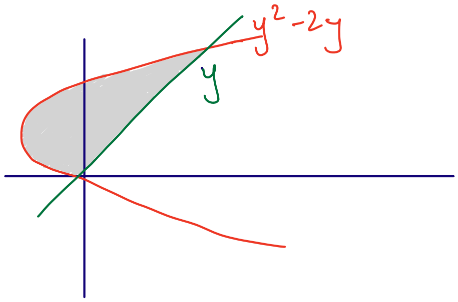
$y = 2 - x \longrightarrow x = 2 - y$

Solving: $\sqrt{y} = 2 - y \quad y = 4 + y^2 - 4y \Rightarrow y^2 - 5y + 4 = 0$

$(y-4)(y-1) = 0 \quad y = 4$ or $y = 1$

$\displaystyle\int_0^1 2 - y - \sqrt{y}\,dy = 5/6$

(d) The finite regions between the curves $x = y^2 - 2y$ and $y = x$.



Solve: $y^2-2y=y$

$y^2-3y=0 \quad y=0$ or $y=3$

Area: $\int_0^3 y - y^2+2y\,dy$

$= 9/2$

ACTIVITY 6.1.4

Each of the following questions somehow involves the arc length along a curve.

(a) Use the definition and appropriate computational technology to determine the arc length along $y = x^2$ from $x = -1$ to $x = 1$.

$f(x) = x^2 \qquad f'(x) = 2x$

Arc length:

$$\int_{-1}^{1} \sqrt{1+(f')^2}\,dx$$

$$= \int_{-1}^{1} \sqrt{1+4x^2}\,dx \approx 2.957$$

(b) Find the arc length of $y = \sqrt{4 - x^2}$ on the interval $-2 \le x \le 2$. Find this value in two different ways, first by using a definite integral, and then by using a familiar property of the curve.

$f(x) = \sqrt{4-x^2} \qquad f'(x) = \frac{1}{\cancel{2}} \frac{-\cancel{2}x}{\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}}$

$$A = \int_{-2}^{2} \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2} = \int_{-2}^{2} \sqrt{1+\frac{x^2}{4-x^2}}\,dx = \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\,dx = 2\int_{-2}^{2} \frac{1}{\sqrt{4-x^2}}\,dx = 2\arcsin\left(\frac{x}{2}\right)\Big|_{-2}^{2}$$

$= 2\pi$

$y = \sqrt{4-x^2} \Rightarrow y^2 = 4-x^2$

$\Rightarrow x^2+y^2=4$ defines a semicircle of radius 2 $\longrightarrow$ Perimeter: $\frac{1}{2}\times 2\pi r = \pi \times 2 = 2\pi$

(c) Determine the arc length of $y = xe^{3x}$ on the interval $[0, 1]$.

$f(x) = xe^{3x}$ $\quad f'(x) = 3xe^{3x} + e^{3x}$ $\quad A = \int_0^1 \sqrt{1 + (3xe^{3x} + e^{3x})^2}$

≈ 20.177

(using numerical integration)

(d) Will the integrals that arise calculating arc length typically be ones that we can evaluate exactly using the first FTC, or ones that we need to approximate? Why?

Typically they're too complicated to evaluate algebraically.

(e) A moving particle is traveling along the curve given by $y = f(x) = 0.1x^2 + 1$, and does so at a constant rate of 7 cm/sec, where both x and y are measured in cm (that is, the curve $y = f(x)$ is the path along which the object actually travels; the curve is not a "position function"). Find the position of the particle when $t = 4$ sec, assuming that when $t = 0$, the particle's location is $(0, f(0))$.

speed = 7 cm/sec

time = 4 sec

total distance = 7×4
$= 28$ cm

$f' = 0.2x$

The final position is unknown let's denote the final x-coordinate by b.

arc-length = total distance

$= \int_0^b \sqrt{1 + 0.04x^2}\, dx = 28$

solving for: $b = 14.9165$