§6.3 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 19 September - 23 September 2022

Density, Mass and Centre of Mass

Main Concepts

- **Total Mass:** For an object of constant cross-sectional area whose mass is distributed along a single axis according to the function $\rho(x)$ (whose units are mass per unit length), the total mass, M, of the object between $x = a$ and $x = b$is given by

$$M = \int_a^b \rho(x)\,\mathrm{d}x$$

- **Centre of mass (for point-masses):** For a collection of n particles $m_1, ..., m_n$ distributed along a line with locations $x_1, ..., x_n$, the centre of mass is given by

$$\overline{x} = \frac{x_1 m_1 + ... + x_n m_n}{m_1 + ... + m_n}$$

- **Centre of mass (for continuous distributions):** For a thin rod of density $\rho(x)$ distributed along the x axis from $x = a$ to $x = b$, the centre of mass of the rod is given by,

$$\overline{x} = \frac{\int_a^b x\rho(x)\,\mathrm{d}x}{\int_a^b \rho(x)\,\mathrm{d}x}$$

Activities

Activity 6.3.2

Consider the following situations in which mass is distributed in a non-constant manner.

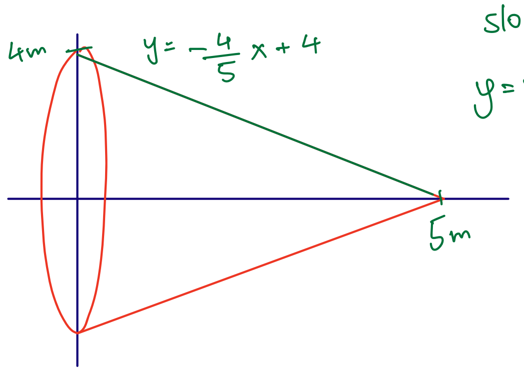
(a) Suppose that a thin rod with constant cross-sectional area of 1 cm^2 has its mass distributed according to the density function $\rho(x) = 2e^{-0.2x}$, where x is the distance in cm from the left end of the rod, and the units on $\rho(x)$ are g/cm. If the rod is 10 cm long, determine the exact mass of the rod.

$$M = \int_0^{10} \rho(x)\,dx = \int_0^{10} 2e^{-0.2x}\,dx = \frac{2}{(-0.2)} e^{-0.2x}\Big|_0^{10}$$

$$= -10\left(e^{-2} - e^{0}\right) = 10\left(1 - e^{-2}\right) \text{ grams}$$

(b) Consider the cone that has a base of radius 4 m and a height of 5 m. Picture the cone lying horizontally with the center of its base at the origin and think of the cone as a solid of revolution.

(i) Write and evaluate a definite integral whose value is the volume of the cone



$\text{slope} = -\frac{4}{5}$ $\quad$ $y\text{-intercept} = 4$

$y = mx + c = -\frac{4}{5}x + 4$

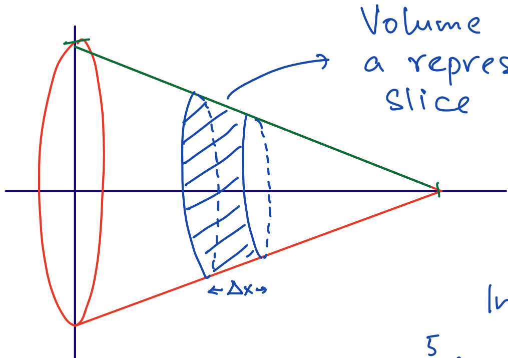
use disk method

$$V = \int_0^5 \pi\left(-\frac{4}{5}x + 4\right)^2 dx = \frac{80\pi}{3}\,\text{m}^3$$

(ii) Next suppose that the cone has uniform density of 800 kg/m^3. What is the mass of the solid cone?

$$M = \rho \times V = 800\,\text{kg/m}^3 \times \frac{80\pi}{3}\,\text{m}^3 = \frac{64000\pi}{3}\,\text{kg}$$

(iii) Now suppose that the cone's density is not uniform, but rather that the cone is most dense at its base. In particular, assume that the density of the cone is uniform across cross sections parallel to its base, but that in each such cross section that is a distance x units from the origin, the density of the cross section is given by the function $\rho(x) = 400 + \frac{200}{1+x^2}$, measured in kg/m^3. Determine and evaluate a definite integral whose value is the mass of this cone of non-uniform density. Do so by first thinking about the mass of a given slice of the cone x units away from the base; remember that in such a slice, the density will be *essentially constant*



Volume of a representative slice $= \pi (r(x))^2 \Delta x = \pi \left(4 - \frac{4}{5}x\right)^2 \Delta x$

Mass of a representative slice $= \rho(x)\left[\pi\left(4-\frac{4}{5}x\right)^2 \Delta x\right]$

Total mass $\approx$ Sum of masses of slices $= \sum \rho(x)\left[\pi\left(4-\frac{4}{5}x\right)^2 \Delta x\right]$

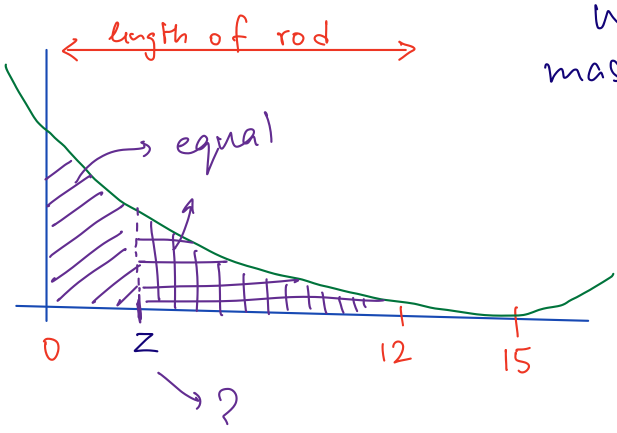
In the limit: $\Delta x \to dx$

$$M = \int_0^5 \rho(x)\, \pi \left(4-\frac{4}{5}x\right)^2 dx = \int_0^5 \left(400+\frac{200}{1+x^2}\right)\cdot \pi \left(4-\frac{4}{5}x\right)^2 dx$$

use a computer

$$= 128\pi\left(\frac{265}{3} + 24\arctan(5) - 5\ln(26)\right)$$

(c) Let a thin rod of constant cross-sectional area 1 cm^2 and length 12 cm have its mass be distributed according to the density function $\rho(x) = \frac{1}{25}(x-15)^2$, measured in g/cm. Find the exact location z at which to cut the bar such that the two pieces will each have identical mass.



Want to find z such that the masses are same on both sides

Mass from 0 to z $= \int_0^z \rho(x)\,dx$

$$= \int_0^z \frac{1}{25}(x-15)^2\,dx$$

$$= \frac{(z-15)^3}{75} - \frac{(0-15)^3}{75} = \frac{(z-15)^3}{75} + 45 \text{ grams}$$

Mass from z to 12: $\int_z^{12} \rho(x)dx = \frac{(12-15)^3}{75} - \frac{(z-15)^3}{75} = \frac{-27}{75} - \frac{(z-15)^3}{75}$

$$= -\frac{9}{25} - \frac{(z-15)^3}{75}$$

Setting the two equal:

$$\frac{(z-15)^3}{75} + 45 = -\frac{9}{25} - \frac{(z-15)^3}{75} \Rightarrow \frac{\cancel{2}^{1}(z-15)^3}{\cancel{75}_{3}} = \frac{\cancel{-1134}^{-567}}{\cancel{25}_{1}} = (z-15)^3 = -567\times 3$$

$$z - 15 = \sqrt[3]{-1701} = -11.93 \Rightarrow z = 15 - 11.937 = 3.063 \text{ cm}$$

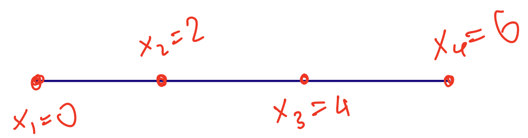
Activity 6.3.3

For quantities of equal weight, such as two children on a teeter-totter, the balancing point is found by taking the average of their locations. When the weights of the quantities differ, we use a weighted average of their respective locations to find the balancing point.

(a) Suppose that a shelf is 6 feet long, with its left end situated at $x = 0$. If one book of weight 1 lb is placed at $x_1 = 0$, and another book of weight 1 lb is placed at $x_2 = 6$, what is the location of $\overline{x}$, the point at which the shelf would (theoretically) balance on a fulcrum?

$$\overline{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{1 \times 0 + 1 \times 6}{1 + 1} = \frac{6}{2} = 3$$

(b) Now, say that we place four books on the shelf, each weighing 1 kb: at $x_1 = 0, x_2 = 2, x_3 = 4$, and $x_4 = 6$. Find $\overline{x}$, the balancing point of the shelf.



$$\overline{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{1 \times 0 + 1 \times 2 + 1 \times 4 + 1 \times 6}{1 + 1 + 1 + 1}$$

$$= \frac{0 + 2 + 4 + 6}{4} = \frac{12}{4} = 3$$

(c) How does $\overline{x}$ change if we change the location of the third book? Say the locations of the 1-lb books are $x_1 = 0, x_2 = 2, x_3 = 3$, and $x_4 = 6$.

$$\overline{x} = \frac{1 \times 0 + 1 \times 2 + 1 \times 3 + 1 \times 6}{4} = \frac{11}{4} = 2.75$$ (shifts to left)

(d) Next, suppose that we place four books on the shelf, but of varying weights: at $x_1 = 0$ a 2-lb book, at $x_2 = 2$ a 3-lb book, at $x_3 = 4$ a 1-lb book and at $x_4 = 6$ a 1-ln book. Use a weighted average of the locations to find $\overline{x}$, the balancing point of the shelf. How does the balancing point in this scenario compare to that found in (b)?

$$\bar{x} = \frac{2\times 0 + 3\times 2 + 1\times 4 + 1\times 6}{2+3+1+1}$$

$$= \frac{6+4+6}{7} = \frac{16}{7} \approx 2.286$$

(e) What happens if we change the location of one of the books? Say that we keep everything the same in (d), except that $x_3 = 5$. How does $\overline{x}$ change?

$$\bar{x} = \frac{2\times 0 + 3\times 2 + 1\times 5 + 1\times 6}{7} = \frac{17}{7} \approx 2.429$$

(shifts to right)

(f) What happens if we change the weight of one of the books? Say that we keep everything the same in (d), except that the book at $x_3 = 4$ now weighs 2 lbs. How does $\overline{x}$ change?

$$\bar{x} = \frac{2\times 0 + 3\times 2 + 2\times 4 + 1\times 6}{2+3+2+1} = \frac{22}{8} = 2.75$$

(shifts to right)

(g) Experiment with a couple of different scenarios of your choosing where you move one of the books to the left, or you decrease the weight of one of the books.

(h) Write a couple of sentences to explain how adjusting the location of one of the books or the weight of one of the books affects the location of the balancing point of the shelf. Think carefully here about how your changes should be considered relative to the location of the balancing point $\overline{x}$ of the current scenario.

Moving a weight left moves the balance point left (and vice-versa if moved right).

If the weight of a book is increased, the balance point shifts closer to that book's location

ACTIVITY 6.3.4

Consider a thin bar of length 20 cm whose density is distributed according to the function $\rho(x) = 4 + 0.1x$, where $x = 0$ represents the left end of the bar. Assume that ρ is measured in g/cm and x is measured in cm.

(a) Find the total mass, M of the bar.

$$M = \int_0^{20} \rho(x)\,dx = \int_0^{20} 4 + 0.1x\,dx = 4x + \frac{0.1x^2}{2}\Big|_0^{20}$$

$$= 100g$$

(b) Without doing any calculations, do you expect the center of mass of the bar to be equal to 10, less than 10, or greater than 10? Why?

The density of the rod increases with x, so there is 'more mass' in the right than the left. We therefore expect the centre of mass to be greater than the midpoint (10cm)

(c) Compute $\overline{x}$, the exact center of mass of the bar.

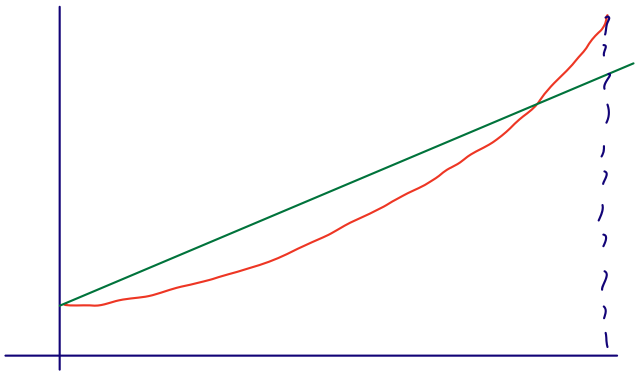
$$\overline{x} = \frac{\int_0^{20} x\rho(x)\,dx}{\int_0^{20} \rho(x)\,dx} = \frac{\int_0^{20} x(4+0.1x)\,dx}{100} = \frac{32}{3}\text{ cm}$$

(d) What is the average density of the bar?

$$\text{Average density} = \frac{\text{total mass}}{\text{total length}}$$

$$= \frac{100g}{20cm} = 5gcm^{-1}$$

(e) Now consider a different density function, given by $p(x) = 4e^{0.020732x}$, also for a bar of length 20 cm whose left end is at $x = 0$. Plot both $\rho(x)$ and $p(x)$ on the same axes. Without doing any calculations, which bar do you expect to have greater center of mass? Why?



Compared to $\rho(x)$, $p(x)$ is "denser" to the right side

so we expect the centre of mass to be greater in this case.

(f) Compute the exact center of mass of the bar described in (e) whose density function is $p(x) = 4e^{0.020732x}$. Check the result against the prediction you made in (e).

$$M = \int_0^{20} p(x)\,dx = \int_0^{20} 4e^{0.020732x}\,dx$$

$$= 99.137$$

$$\bar{x} = \frac{\int_0^{20} x\,p(x)}{\int_0^{20} p(x)} = \frac{\int_0^{20} x\,4e^{0.020732x}\,dx}{99.137}$$

$$= 10.69 \text{ cm}$$

It is indeed greater than the centre of mass for $\rho(x)$.