§6.4 Name: SOLUTIONS

Fall MATH 1120 Lec003 26 September - 30 September 2022

Work, Force and Pressure

Main Concepts

- **Work:** The work done to move an object on the x-axis from a to b by a force $F(x)$ is given by

$$W = \int_a^b F(x)\,\mathrm{d}x$$

- **Pumping Liquid from a Tank:**
  1. Determine the volume of a slice of thickness Δx at location x, $V_{\text{slice}} = A_{\text{slice}}(x)\Delta x$.
  2. Find the weight (force), $F_{\text{slice}}(x) = \rho V_{\text{slice}}$ of that slice (remember to multiply by acceleration due to gravity, $g = 9.81m/s^2$ if using metric units!).
  3. Determine the distance, $d_{\text{slice}}(x)$ that the slice a x has to move.
  4. The work done to move the slice is then given by $W_{\text{slice}} = F_{\text{slice}} d_{\text{slice}} = \rho A_{\text{slice}} \Delta x d_{\text{slice}}$.
  5. The total work done to move the entire liquid is then given by integrating the over the slices ($\Delta x \to \mathrm{d}x$), i.e., $W = \int \rho A_{\text{slice}} d_{\text{slice}}\,\mathrm{d}x$ and the limits of integration are the x limits of the tank.

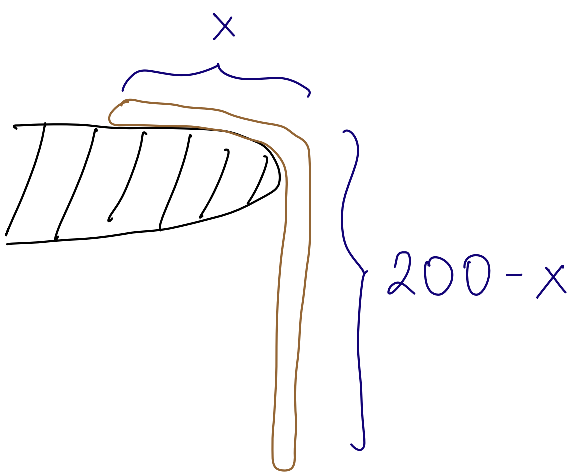
- **Pressure:** $P = F/A$ and $F = P \cdot A$.

Activities

Activity 6.4.2

Consider the following situations in which a varying force accomplishes work.

(a) Suppose that a heavy rope hangs over the side of a cliff. The rope is 200 feet long and weighs 0.3 pounds per foot; initially the rope is fully extended. How much work is required to haul in the entire length of the rope? (**Hint:** set up a function $F(h)$ whose value is the weight of the rope remaining over the cliff after h feet have been hauled in.)



Suppose x is length of rope that's been hauled in already. The weight of the remaining rope is then $0.3 \times (200-x)$

Work $= \int_0^{200} 0.3(200-x)\,dx = 6000$ ftlb

to haul the entire rope

(b) A leaky bucket is being hauled up from a 100 foot deep well. When lifted from the water, the bucket and water together weigh 40 pounds. As the bucket is being hauled upward at a constant rate, the bucket leaks water at a constant rate so that it is losing weight at a rate of 0.1 pounds per foot. What function $B(h)$ tells the weight of the bucket after the bucket has been lifted h feet? What is the total amount of work accomplished in lifting the bucket to the top of the well?

The picture is very similar to the previous one. The weight of the bucket once it's been lifted h feet is

$B(h) = 40 - 0.1h$ So the work done is $\int_0^{100} 40 - 0.1h\,dh = 3500$ ft lb

(c) Now suppose that the bucket in (b) does not leak at a constant rate, but rather that its weight at height h feet above the water is given by $B(h) = 25 + 15e^{-0.05h}$. What is the total work required to lift the bucket 100 feet? What is the average force exerted on the bucket on the interval $h = 0$ to $h = 100$?

The total work done is

$$W = \int_0^{100} 25 + 15e^{-0.05h} = 2797.98 \text{ ft lb}$$

The average force is the above divided by the length of the interval so 27.98 lb.

(d) From physics, Hooke's Law for springs states that the amount of force required to hold a spring that is compressed (or extended) to a particular length is proportionate to the distance the spring is compressed (or extended) from its natural length. That is, the force to compress (or extend) a spring x units from its natural length is $F(x) = kx$ for some constant k (which is called the spring constant.) For springs, we choose to measure the force in pounds and the distance the spring is compressed in feet. Suppose that a force of 5 pounds extends a particular spring 4 inches (1/3 foot) beyond its natural length.

(i) Use the given fact that $F(1/3) = 5$ to find the spring constant k.

$F(x) = kx$ $\quad\quad F(1/3) = 5 = k \cdot \frac{1}{3}$

$\Rightarrow k = 15$

(ii) Find the work done to extend the spring from its natural length to 1 foot beyond its natural length.

$$W = \int F(x)\,dx = \int_0^1 15x\,dx = \frac{15}{2} \text{ ft lb}$$

(iii) Find the work required to extend the spring from 1 foot beyond its natural length to 1.5 feet beyond its natural length

$$W = \int_{1}^{1.5} 15x \, dx = 9.375 \text{ ft lb}$$

ACTIVITY 6.4.3

In each of the following problems, determine the total work required to accomplish the described task. In parts (b) and (c), a key step is to find a formula for a function that describes the curve that forms the side boundary of the tank.

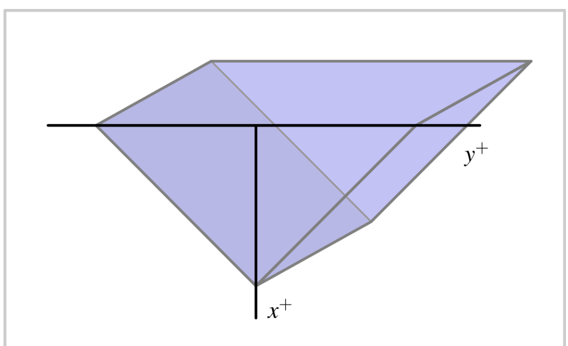
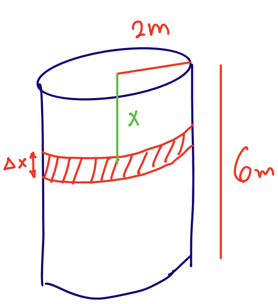


Figure 1: A trough with triangular ends

(a) Consider a vertical cylindrical tank of radius 2 meters and depth 6 meters. Suppose the tank is filled with 4 meters of water of mass density 1000 kg/m^3, and the top 1 meter of water is pumped over the top of the tank.



$$V_{slice} = A_{slice}\,\Delta x$$
$$= \pi (2)^2 \Delta x = 4\pi \Delta x$$

Weight of slice $= \rho \cdot g \cdot V_{slice}$
$$= 1000 \times 9.81 \times 4\pi \Delta x$$

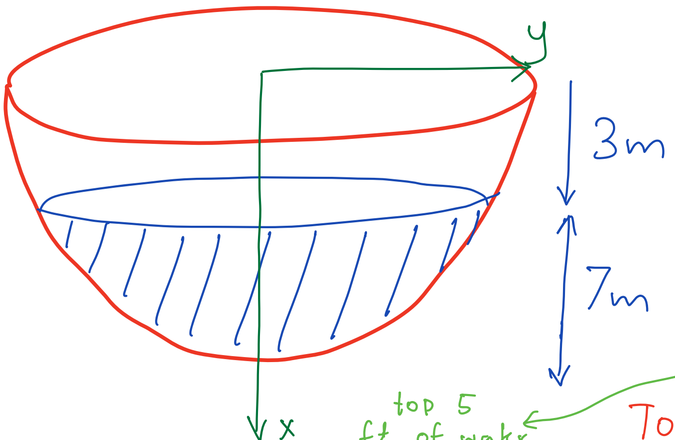
Water from the top 1 m is pumped out. So x goes from $x = 2$ to $x = 3$

$$W = \int_{2}^{3} 1000 \times 9.81 \times 4\pi \; x \, dx$$
$$= 308190 \text{ Nm}$$

(b) Consider a hemispherical tank with a radius of 10 feet. Suppose that the tank is full to a depth of 7 feet with water of weight density 62.4 pounds/ft^3, and the top 5 feet of water are pumped out of the tank to a tanker truck whose height is 5 feet above the top of the tank.

$$\text{Volume of a slice} = \underbrace{\pi\left(\sqrt{1-x^2}\right)^2}_{\text{as a solid of revolution}} \Delta x$$

$$= \pi(1-x^2)\,\Delta x$$



$$\text{Weight of a slice} = 62.4 \times \pi(1-x^2)\Delta x$$

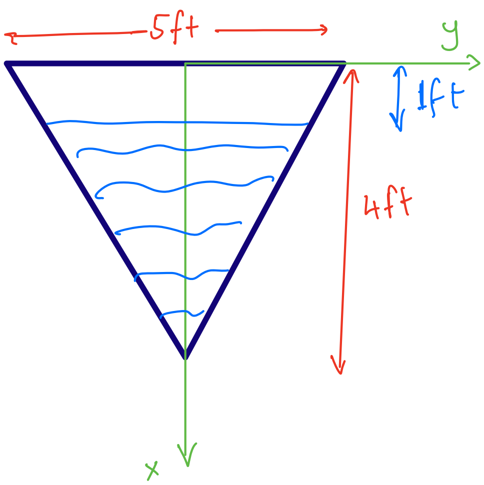
Since the water is pumped a distance 5 ft above the top,

$$W_{slice} = \text{Weight of Slice} \times (x+5)$$

top 5 ft of water is pumped out

$$\text{Total work} = \int_3^8 62.4\pi(100-x^2)(x+5)\,dx \approx 673593 \text{ ftlb}$$

(c) Consider a trough with triangular ends, as pictured in Figure 1, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft^3, and a pump is used to empty the tank until the water remaining in the tank is 1 foot deep.



Equation of line: Slope $= -\frac{5}{2}/4 = -\frac{5}{8}$

y-intercept $= 5/2$ $\quad y = -\frac{5}{8}x + \frac{5}{2}$

$$\text{Vol of a Slice} = 2\cdot\left(-\frac{5}{8}x+\frac{5}{2}\right)\cdot 10\cdot\Delta x$$

$$\text{Weight of Slice} = 62.4 \times \text{Vol of Slice}$$

Work to move one slice by distance x:

$$W_{slice} = 62.4 \cdot 20 \cdot \left(-\frac{5}{8}x+\frac{5}{2}\right)\Delta x \cdot x$$

$$\text{Total work} = \int_1^3 62.4\cdot 20\cdot\left(-\frac{5}{8}x+\frac{5}{2}\right)x\,dx \approx 5720 \text{ftlb}$$

Activity 6.4.4

In each of the following problems, determine the total force exerted by water against the surface that is described.

(a) Consider a rectangular dam that is 100 feet wide and 50 feet tall, and suppose that water presses against the dam all the way to the top.

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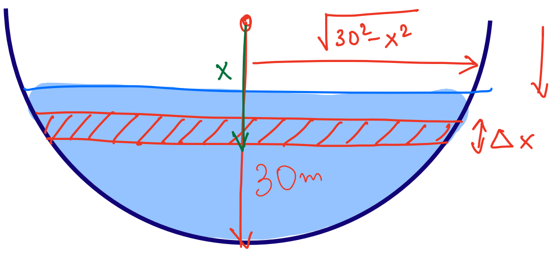
(b) Consider a semicircular dam with a radius of 30 feet. Suppose that the water rises to within 10 feet of the top of the dam.

$$A_{slice} = 2\sqrt{30^2 - x^2}\,\Delta x$$



Force experienced by slice a depth x: $62.4 \cdot A_{slice} \cdot (x-10)$

because the water lies 10ft below the top

Total Force: $\int_{10}^{30} 2\sqrt{30^2 - x^2} \times 62.4 \times (x-10)\,dx$

≈ 800244 lb

(c) Consider a trough with triangular ends, as pictured in Figure 1, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft^3. How much force does the water exert against one of the triangular ends?

Equation of line: $y = -\frac{5}{8}x + \frac{5}{2}$. We only care about the triangular sides

Area of a Slice: $2 \cdot \left(-\frac{5}{8}x + \frac{5}{2}\right) \cdot \Delta x$

water is 1ft below the top of the tank

Force acting on a slice $= 62.4 \cdot A_{slice} \cdot (x-1)$

Total force $= \int_1^4 62.4 \cdot 2 \cdot \left(-\frac{5}{8}x + \frac{5}{2}\right)(x-1)\,dx = 351$ pounds

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(a) $A_{slice} = 100 \cdot \Delta x$ $\quad F_{slice} = 100 \cdot \Delta x \cdot x \cdot 62.4$

$$F_{total} = \int_0^{50} 6240\,x\,dx = 7800000 \text{ lb}$$