§6.5 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 03 September - 07 September 2022

Improper Integrals

Main Concepts

- Integrals can be used to represent the likelihood of certain events.
- **Improper Integrals on Unbounded Intervals:** Integrals over unbounded domains can be defined as the limit of definite integrals on bounded domains. For example,

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x.$$

- **Convergence and Divergence:** The limit given above may not always exist. If the limit $\int_a^\infty f(x)\,\mathrm{d}x$ exists, we say that the improper integral converges otherwise we say it diverges.
- **Improper Integrals for Unbounded Integrands:** Sometimes a function may be unbounded on an interval, but its integral may still be defined by using a limit. For example, $1/\sqrt{x}$ blows up to infinity as $x \to 0^+$, however we can integrate it over $[0,1]$ by defining:

$$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{x\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x.$$

  We can also talk about the convergence and divergence of such integrals similar to the previous point.

- There are instances where applying the fundamental theorem without care might lead to erroneous results, for example when the function is not continuous (or not defined) on parts of the interval: $\int_{-1}^1 \frac{1}{x^2}\,\mathrm{d}x$. These can be treated as improper integrals in the following way:

$$\int_{-1}^1 \frac{1}{x^2}\,\mathrm{d}x = \lim_{a\to 0^-}\int_{-1}^a \frac{1}{x^2}\,\mathrm{d}x + \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}\,\mathrm{d}x = \infty.$$

- An important class of improper integrals is $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$, where $p > 0$ is a real number. The integral converges for $p > 1$ and diverges for $p \le 1$.

  Another important class is $\int_0^1 \frac{1}{x^p}\,\mathrm{d}x$ which diverges for $p \ge 1$ but converges for $p < 1$.

Activities

Activity 6.5.2

In this activity we explore the improper integrals $\int_1^\infty \frac{1}{x}\,dx$ and $\int_1^\infty \frac{1}{x^{3/2}}\,dx$.

(a) First we investigate $\int_1^\infty \frac{1}{x}\,dx$.

(i) Use the first FTC to determine the exact values of $\int_1^{10} \frac{1}{x}\,dx$, $\int_1^{1000} \frac{1}{x}\,dx$, and $\int_1^{100000} \frac{1}{x}\,dx$. Then, use your computational device to compute a decimal approximation of each result.

$\int_1^{10} \frac{1}{x}\,dx = \ln 10$; $\int_1^{1000} \frac{1}{x}\,dx = \ln 1000$; $\int_1^{100000} \frac{1}{x}\,dx = \ln 100000$

$= 2.30$ $\quad = 6.91$ $\quad = 11.51$

(ii) Use the first FTC to evaluate the definite integral $\int_1^b \frac{1}{x}\,dx$ (which results in an expression that depends on b).

$\int_1^a \frac{1}{x}\,dx = \ln x \Big|_1^a = \ln a$

(iii) Now, use your work from (ii) to evaluate the limit given by

$$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx.$$

$\lim_{b\to\infty} \ln b = \infty$

(b) Next we investigate $\int_1^\infty \frac{1}{x^{3/2}}\,dx$.

(i) Use the first FTC to determine the exact values of $\int_1^{10} \frac{1}{x^{3/2}}\,dx$, $\int_1^{1000} \frac{1}{x^{3/2}}\,dx$, and $\int_1^{100000} \frac{1}{x^{3/2}}\,dx$. Then, use your computational device to compute a decimal approximation of each result.

$\int_1^{10} \frac{1}{x^{3/2}}\,dx = \frac{-2}{\sqrt{10}} + 2$; $\int_1^{1000} \frac{1}{x^{3/2}}\,dx = \frac{-2}{\sqrt{1000}} + 2$; $\int_1^{100000} \frac{1}{x^{3/2}}\,dx = \frac{-2}{\sqrt{100000}} + 2$

$= 1.37$ $\qquad = 1.94$ $\qquad = 1.99$

(ii) Use the first FTC to evaluate the definite integral $\int_1^b \frac{1}{x^{3/2}}\,dx$ (which results in an expression that depends on b).

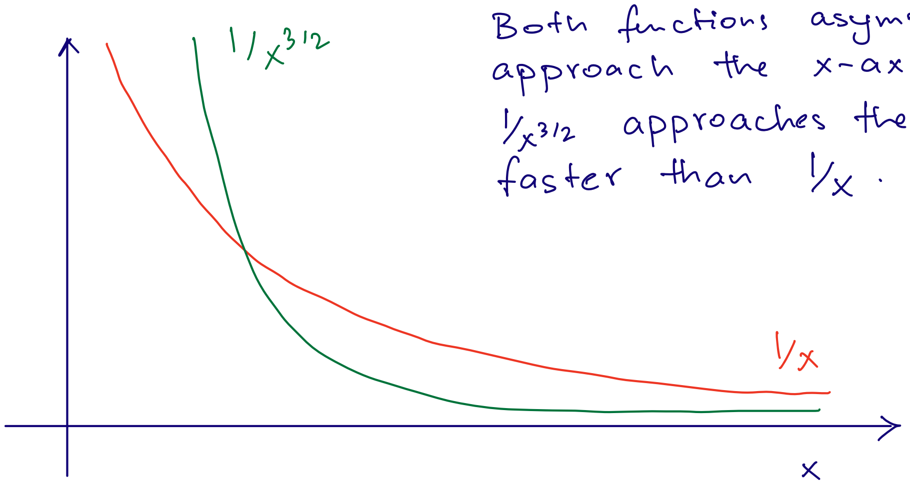
$\int_1^b \frac{1}{x^{3/2}}\,dx = -\frac{2}{x^{1/2}}\Big|_1^b = \frac{-2}{\sqrt{b}} + 2$

(iii) Now, use your work from (ii) to evaluate the limit given by

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^{3/2}}\,dx.$$

$\lim_{b\to\infty} \frac{-2}{\sqrt{b}} + 2 = 2$

(c) Plot the functions $y = \frac{1}{x}$ and $y = \frac{1}{x^{3/2}}$ on the same coordinate axes. How would you compare their behavior as x increases without bound? What is similar? What is different?



Both functions asymptotically approach the x-axis. $1/x^{3/2}$ approaches the x-axis faster than $1/x$.

(d) How would you characterize the value of $\int_1^\infty \frac{1}{x}$? Of $\int_1^\infty \frac{1}{x^{3/2}}$? What does this tell us about the respective areas bounded by these two curves for $x \geq 1$?

$\int_1^\infty \frac{1}{x}\,dx$ diverges whereas $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges.

The area bounded by $1/x$ is infinite while the area bounded by $1/x^{3/2}$ is finite on the region $[1,\infty)$

Activity 6.5.3

Determine whether each of the following improper integrals converges or diverges. For each integral that converges, find its exact value.

(a) $\int_1^\infty \frac{1}{x^2}\,dx$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{b} + 1$$

$$= 1 \quad \text{(convergent)}$$

(b) $\int_0^\infty e^{-x/4}\,dx$

$$\lim_{b\to\infty} \int_0^b e^{-x/4}\,dx = \lim_{b\to\infty} -4e^{-x/4}\Big|_0^b$$

(convergent)

$$= \lim_{b\to\infty} -4\left(e^{-b/4} - e^{0}\right)$$

$$= \lim_{b\to\infty} -4\left(e^{-b/4} - 1\right)$$

$$= 4$$

$$\left(\lim_{b\to\infty} e^{-b/4} = 0\right)$$

(c) $\int_2^\infty \frac{9}{(x+5)^{2/3}}\,dx$

$$\lim_{b\to\infty} \int_2^b \frac{9}{(x+5)^{2/3}}\,dx$$

let $(x+5) = u$ $\quad du = dx$

limits: $2+5=7$ to $2+b$

$$\lim_{b\to\infty} \int_7^{b+2} \frac{9}{u^{2/3}}\,du = \lim_{b\to\infty} 3u^{1/3}\Big|_7^{b+2}$$

(divergent)

$$= \lim_{b\to\infty} 3(b+2)^{1/3} - 3(7)^{1/3}$$

$$= \infty$$

(d) $\int_4^\infty \frac{3}{(x+2)^{5/4}}\,dx$

$$\lim_{b\to\infty} \int_4^b \frac{3}{(x+2)^{5/4}}\,dx = \lim_{b\to\infty} \frac{3\,(x+2)^{-\frac{1}{4}}}{\left(-\frac{1}{4}\right)}\Bigg|_4^b = \lim_{b\to\infty} \frac{-12}{(b+2)^{1/4}} + \frac{12}{(6)^{1/4}}$$

(convergent)

$$= \frac{12}{6^{1/4}}$$

(e) $\int_0^\infty xe^{-x/4}\,dx$

Integrate by parts!

$$\lim_{b\to\infty}\int_0^b xe^{-x/4}\,dx = \lim_{b\to\infty}\left[-4xe^{-x/4}\Big|_0^b + 4\int_0^b e^{-x/4}\,dx\right]$$

$$= \lim_{b\to\infty}\left[-4\left(be^{-b/4} - 0\cdot e^{-0/4}\right) + 4(-4)e^{-x/4}\Big|_0^b\right]$$

$$= \lim_{b\to\infty}\left[-4be^{-b/4} - 16\left(e^{-b/4} - 1\right)\right]$$

$$\left(\lim_{b\to\infty} be^{-b/4} = \lim_{b\to\infty}\frac{b}{e^{b/4}} = \lim_{b\to\infty}\frac{\frac{d}{db}b}{\frac{d}{db}e^{b/4}} = \lim_{b\to\infty}\frac{1}{\frac{1}{4}e^{b/4}} = 0 \text{ (L'Hôpital's rule)}\right)$$

$$= 16$$

(f) $\int_1^\infty \frac{1}{x^p}\,dx$, where p is a positive real number

$$\int_1^b x^{-p}\,dx = \frac{x^{-p+1}}{-p+1}\Big|_1^b = \frac{b^{-p+1} - 1^{-p+1}}{-p+1} = \frac{b^{1-p} - 1}{1-p}$$

This formula only works for $p \neq 1$ so we need to handle the $p=1$ case separately

$$\int_1^\infty x^{-p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\frac{b^{1-p}}{1-p} - \frac{1}{1-p}$$

if $1-p > 1$, i.e. $1 > p$, the limit doesn't exist (divergent)

if $1-p < 1$, i.e. $1 < p$, the limit exists (convergent)

$p=1$ case: $\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b = \infty$

Conclusion:

$0 < p \leq 1 \longrightarrow$ Divergent

$p > 1 \longrightarrow$ Convergent

Activity 6.4.4

For each of the following definite integrals, decide whether the integral is improper or not. If the integral is proper, evaluate it using the First FTC. If the integral is improper, determine whether or not the integral converges or diverges; if the integral converges, find its exact value.

(a) $\int_0^1 \frac{1}{x^{1/3}}\,dx$

$$\lim_{b\to 0^+}\int_b^1 \frac{1}{x^{1/3}}\,dx = \lim_{b\to 0^+}\frac{x^{2/3}}{2/3}\Big|_b^1 = \lim_{b\to 0^+}\frac{3}{2}\left(1 - b^{2/3}\right) = \frac{3}{2}$$

(convergent)

(b) $\int_0^2 e^{-x}\,dx$

This is just a proper integral

$$\int_0^2 e^{-x}dx = -e^{-x}\Big|_0^2 = -(e^{-2}-e^{0})$$

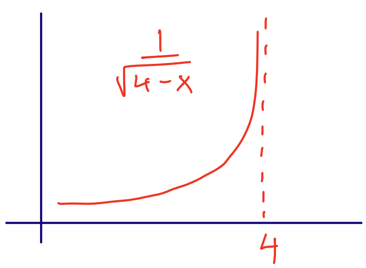
$$= 1-e^{-2}.$$

(c) $\int_1^4 \frac{1}{\sqrt{4-x}}\,dx$

$$\lim_{b\to 4^-}\int_1^b \frac{1}{\sqrt{4-x}}\,dx = \lim_{b\to 4^-} -\frac{(4-x)^{1/2}}{(1/2)}\Big|_1^b$$

$$= \lim_{b\to 4^-}\left[2\cdot 3^{1/2} - 2\cdot(4-b)^{1/2}\right]$$

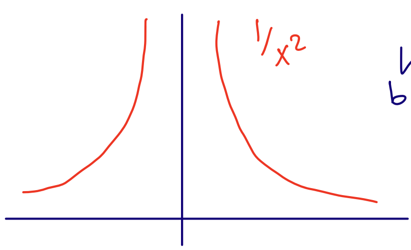
$$= 2\cdot 3^{1/2} \quad \text{(convergent)}$$



This function "blows-up" at $x=4$, so we take the limit $b\to 4^-$.

(d) $\int_{-2}^2 \frac{1}{x^2}\,dx$

Split into two integrals



$$\lim_{b\to 0^-}\int_{-2}^b \frac{1}{x^2}dx + \lim_{a\to 0^+}\int_a^2 \frac{1}{x^2}dx$$

$$= \lim_{b\to 0^-} \frac{-x^{-2+1}}{}\Big|_{-2}^b + \lim_{a\to 0^+} -x^{-2+1}\Big|_a^2$$

$$= \lim_{b\to 0^-}\left[-\frac{1}{b}-\frac{1}{2}\right] + \lim_{a\to 0^+}\left[-\frac{1}{2}+\frac{1}{a}\right] = \infty \quad \text{(divergent)}$$

(e) $\int_0^{\pi/2} \tan(x)\, dx$

$$\lim_{b \to \frac{\pi}{2}} \int_0^b \tan x\, dx = \lim_{b \to \frac{\pi}{2}} \ln|\sec x| \Big|_0^b = \lim_{b \to \frac{\pi}{2}} \left[\ln|\sec b| - 1\right]$$

as $b \to \frac{\pi}{2}$ $\sec b \to \infty$ so $\ln|\sec b| \to \infty$

thus the integral is divergent

(f) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$

for $0 \leq t \leq \pi/2$

let $x = \sin t$ $\quad dx = \cos t\, dt \quad \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \cos t$

$$\lim_{b \to \pi/2} \int_0^b \frac{\cancel{\cos t}\, dt}{\cancel{\cos t}} = \int_0^{\pi/2} dt = \pi/2$$

The integral converges