§7.1 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 10 October - 14 October 2022

Differential Equations

Main Concepts

- **Differential Equation:** An equation relating an *unknown* function, $y(t)$ and its derivatives, $\frac{\mathrm{d}y}{\mathrm{d}t}$, $\frac{\mathrm{d}y}{\mathrm{d}t}$, etc.
- The variable y representing the unknown function is referred to as the *dependent* variable and t is referred to as the *independent* variable.
- The **Order** of a Differential Equation is the order of the highest derivative appearing in the equation. For example,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 + \sin(y) \text{ is 1st order and,}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^3 + y \text{ is 2nd order.}$$

- A differential equation is called **Autonomous** if the independent variable does not appear explicitly in the equation.
- **Solution of a Differential equation:** A function that satisfies the relation described by the differential equation is called a *solution* of the differential equation.
- To check if a function is a solution to a given differential equation, we "plug in" the function into the left and right hand sides of the equation and check if the they are equal.

Activities

Activity 7.1.2

Express the following statements as differential equations. In each case, you will need to introduce notation to describe the important quantities in the statement so be sure to clearly state what your notation means.

(a) The population of a town that grows continuously at an annual rate of 1.25%.

$$\frac{dP}{dt} = \frac{1.25}{100} P$$

where

$P(t)$: Population at time t

(b) A radioactive sample loses mass at a rate of 5.6% of its mass every day.

$$\frac{dm}{dt} = -\frac{5.6}{100} m$$

where

$m(t)$: mass at time t

negative sign because mass is lost

(c) You have a bank account that continuously earns 0.13% interest every year. At the same time, you withdraw money continually from the account at the rate of 1000 per year.

$$\frac{dM}{dt} = \underbrace{\frac{0.13}{100} M}_{\text{rate of increase depends on your balance}} - \underbrace{1000}_{\text{rate of decrease is constant}}$$

$M(t)$: Money in your account

(d) A cup of hot chocolate is sitting in a 70° room. The temperature of the hot chocolate cools continuously by 10% of the difference between the hot chocolate's temperature and the room temperature every minute.

$$\frac{dT}{dt} = -0.1(T-70)$$

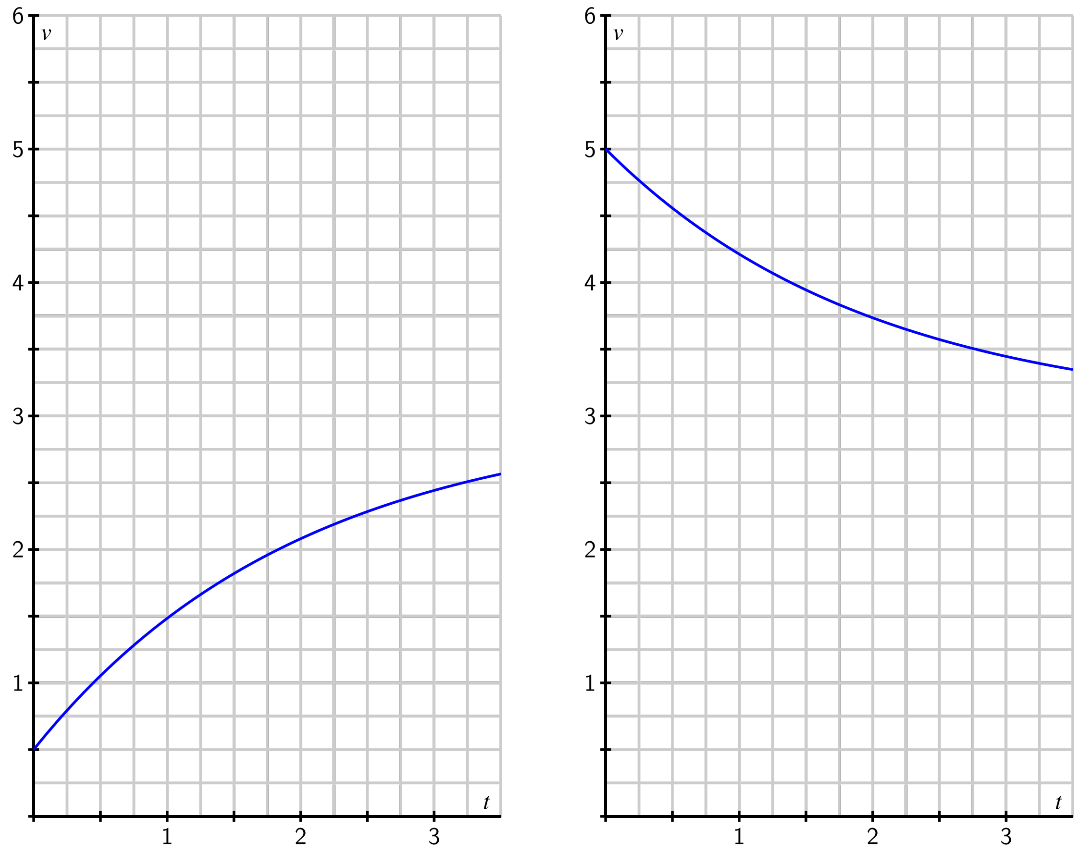
$T(t)$: temperature @ time t

(e) A can of cold soda is sitting in a 70° room. The temperature of the soda warms continuously at the rate of 10% of the difference between the soda's temperature and the room's temperature every minute.

$$\frac{dT}{dt} = 0.1(70-T) = -0.1(T-70)$$

Activity 7.1.3

Figure 1 shows two graphs depicting the velocity of falling objects. On the left is the velocity of a skydiver, while on the right is the velocity of a meteorite entering the Earth's atmosphere.



(a) A skydiver's velocity

(b) A meteorite's velocity

Figure 1: Velocities

(a) Begin with the skydiver's velocity and use the given graph to measure the rate of change dv/dt when the velocity is $v = 0.5, 1.0, 1.5$, and 2.5. Plot the values on Figure 2 You will want to think carefully about this: you are plotting the derivative dv/dt as a function of *velocity*.

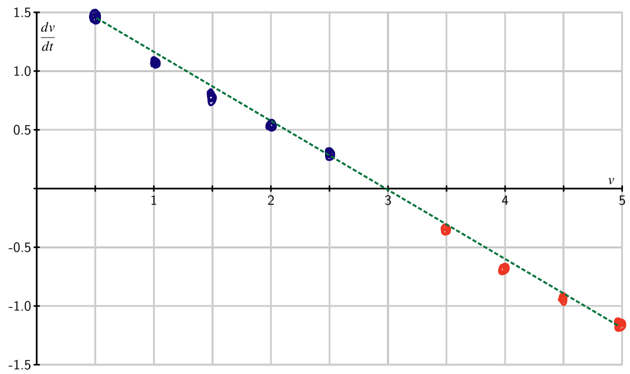


Figure 2: Acceleration as a function of velocity

(b) Now do the same thing with the meteorite's velocity: use the given graph to measure the rate of change dv/dt when the velocity is $v = 3.5, 4.0, 4.5$, and 5.0. Plot your values on Figure 2.

(c) You should find that all your points lie on a line. Write the equation of this line being careful to use proper notation for the quantities on the horizontal and vertical axes.

$$\frac{dv}{dt} = -\frac{1}{2}v + \frac{3}{2}$$

(your answer would vary depending on the approximations you made)

(d) The relationship you just found is a differential equation. Write a complete sentence that explains its meaning.

It says that rate of change of velocity, (a.k.a acceleration) is negatively proportional to velocity. For low velocity, the acceleration is positive, i.e. it speeds up, but after a certain point, acceleration becomes negative, i.e. it slows down.

(e) By looking at the differential equation, determine the values of the velocity for which the velocity increases.

$\frac{dv}{dt} > 0$: $-\frac{1}{2}v + \frac{3}{2} > 0 \Rightarrow v < 3$

↳ positive acceleration

(f) By looking at the differential equation, determine the values of the velocity for which the velocity decreases.

$\frac{dv}{dt} < 0$: $-\frac{1}{2}v + \frac{3}{2} < 0 \Rightarrow v > 3$

↳ negative acceleration (deceleration)

(g) By looking at the differential equation, determine the values of the velocity for which the velocity remains constant.

When $v=3$ $\frac{dv}{dt} = -\frac{1}{2} \times 3 + 3 = 0$. So the velocity is not changing.

Activity 7.1.4

Consider the differential equation

$$\frac{dv}{dt} = 1.5 - 0.5v.$$

which of the following functions are solutions of this differential equation?

(a) $v(t) = 1.5t - 0.25t^2$.

LHS: $\frac{dv}{dt} = 1.5 - 0.5t$

RHS: $1.5 - 0.5v = 1.5 - 0.5(1.5t - 0.25t^2)$

$= 1.5 - 0.75t + 0.0125t^2$

LHS $\neq$ RHS $\Rightarrow$ Not a solution

(b) $v(t) = 3 + 2e^{-0.5t}$.

LHS: $\frac{dv}{dt} = -e^{-0.5t}$

RHS: $1.5 - 0.5v = 1.5 - 0.5(3 + 2e^{-0.5t})$

$= 1.5 - 1.5 - e^{-0.5t} = -e^{-0.5t}$

LHS = RHS $\Rightarrow$ This is a solution

(c) $v(t) = 3$.

LHS: $\frac{dv}{dt} = 0$

RHS: $1.5 - 0.5v = 1.5 - 0.5 * 3 = 0$

LHS = RHS $\Rightarrow$ This is a solution

(d) $v(t) = 3 + Ce^{-0.5t}$, where C is any constant.

LHS: $\frac{dv}{dt} = -0.5Ce^{-0.5t}$

RHS: $1.5 - 0.5v = 1.5 - 0.5(3 + Ce^{-0.5t})$

$= 1.5 - 1.5 - 0.5Ce^{-0.5t}$

$= -0.5Ce^{-0.5t}$

LHS = RHS $\Rightarrow$ This is a solution.