§7.2 Name: SOLUTION

Fall MATH 1120 Lec003 17 October - 21 October 2022

Qualitative Behaviour of Solutions to Differential Equations

Main Concepts

- Observe that a differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t)$ describes the slope of y (i.e. its derivative) as a function of y and t.
- A **Slope Field** is a obtained by graphing the slope or tangent line at different points on the t-y plane.
- Once a slope field is obtained, the graph of a solution can be plotted by drawing a curve that is tangent to the lines of the slope field. This would amount to starting at some point in the t-y plane and "following the arrows" given by the slope field.
- Autonomous differential equations, $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$ can sometimes have solutions that are constant in time. These are known as **equilibrium solutions**.
- Equilibrium solutions are obtained by solving the equation $f(y) = 0$.
- An equilibrium solution is *stable* if nearby solutions are "pulled in" as time progresses.
- An equilibrium solution is *unstable* if nearby solutions are "pushed away" as time progresses.

Activities

Activity 7.2.2

Consider the autonomous differential equation

$$\frac{dy}{dt} = -\frac{1}{2}(y-4).$$

(a) Make a plot of $\frac{dy}{dt}$ versus y on the axes on Figure 1. Looking at the graph, for what values of y does y increase and for what values of y does y decrease?

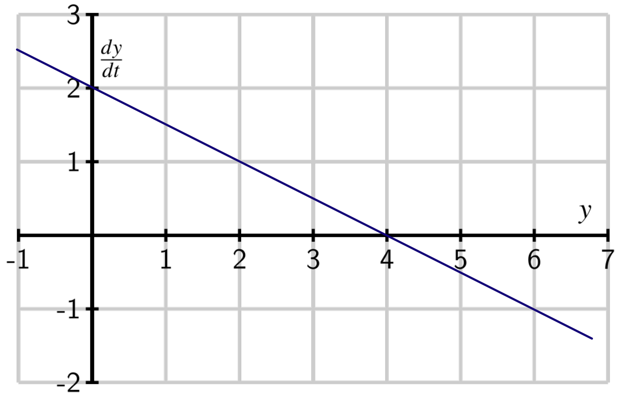

Figure 1: Axes for plotting $\frac{dy}{dt}$ versus y

(b) Next, sketch the slope field for this differential equation on the axes on Figure 2

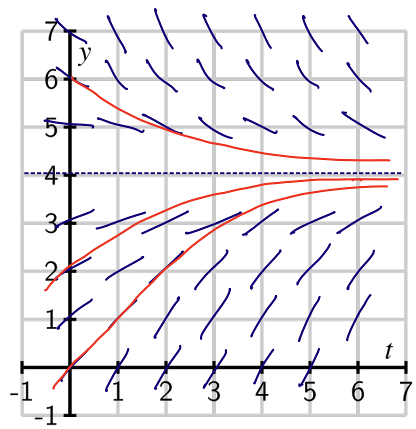

Figure 2: Axes for plotting the slope field

(c) Use your work in (b) to sketch (on the same axes as in Figure 2) solutions that satisfy $y(0) = 0, y(0) = 2$, and $y(0) = 6$.

(d) Verify that $y(t) = 4 + 2e^{-t/2}$ is a solution to the given differential equation with the initial values $y(0) = 6$. Compare its graph to the one you sketched in (c).

LHS: $\frac{dy}{dt} = 2\left(-\frac{1}{2}\right)e^{-t/2} = -e^{-t/2}$

RHS: $-\frac{1}{2}(y-4) = -\frac{1}{2}(4+2e^{-t/2}-4) = -\frac{1}{2}(2e^{-t/2}) = -e^{-t/2}$

$y(0) = 4 + 2e^0$
$= 6$

(e) What is special about the solution where $y(0) = 4$?

the solution is the constant function.

ACTIVITY 7.2.3

Consider the autonomous differential equation

$$\frac{dy}{dt} = -\frac{1}{2}y(y-4).$$

(a) Make a plot of $\frac{dy}{dt}$ versus y on the axes provided in Figure 3. Looking at the graph, for what values of y does y increase and for what values of y does y decrease?

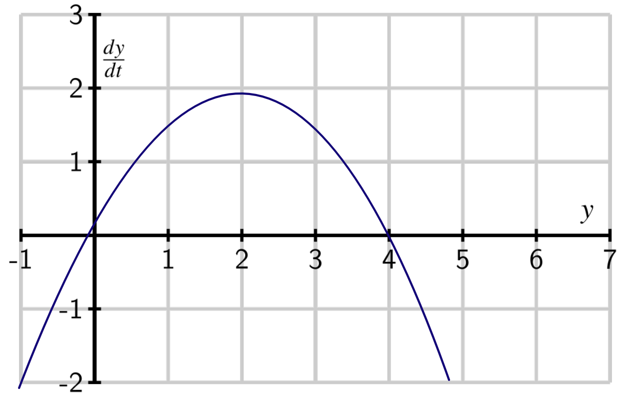

Figure 3: Axes for plotting $\frac{dy}{dt}$ versus y

(b) Identify any equilibrium solutions of the given differential equation.

$y = 0$ and $y = 4$

(c) Now sketch the slope field for the given differential equation on the axes provided in Figure 4

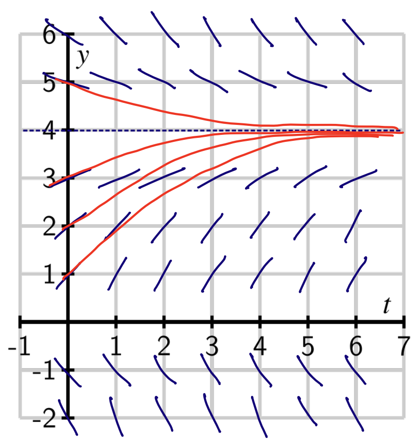

Figure 4: Axes for plotting the slope field

(d) Sketch the solutions to the given differential equation that correspond to initial values $y(0) = -1, 0, 1, \ldots, 5$.

(e) An equilibrium solution $\overline{y}$ is called **stable** if nearby solutions converge to $\overline{y}$. This means that if the initial condition varies slightly from $\overline{y}$, then $\lim_{t\to\infty} y(t) = \overline{y}$. Conversely, an equilibrium solution $\overline{y}$ is called **unstable** if nearby solutions are pushed away from $\overline{y}$. Using your work above, classify the equilibrium solutions you found in (b) as either stable or unstable.

$y = 4$ is stable

$y = 0$ is unstable

(f) Suppose that $y(t)$ described the population of a species of living organisms and that the initial value $y(0)$ is positive. What can you say about the eventual fate of this population?

The population will tend to 4 eventually

(g) Now consider a general autonomous differential equation of the form $dy/dt = f(y)$. Remember that an equilibrium solution $\overline{y}$ satisfies $f(\overline{y}) = 0$. If we graph $dy/dt = f(y)$ as a function of y, for which of the differential equations represented in Figure 5 is $\overline{y}$ a stable equilibrium and for which is $\overline{y}$ unstable? Why?

unstable

stable

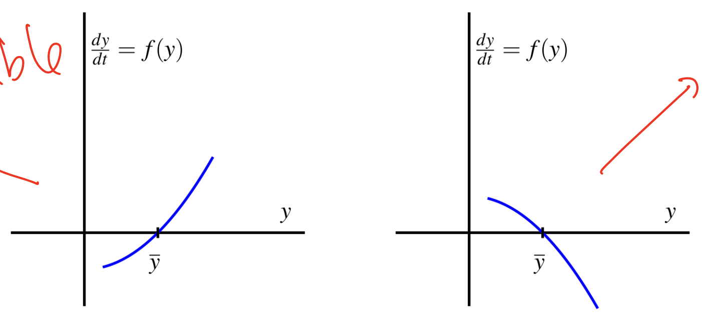

Figure 5: Plots of $\frac{dy}{dt}$ as a function of y