§7.3 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 17 October - 21 October 2022

Euler's Method

Main Concepts

- **Euler's method** is a technique/algorithm used to approximate the solution to a differential equation.
- Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t)$ and suppose we know that at $t = t_0$ $y(t_0) = y_0$. It's useful to think of y as position and t as time (although this may not always be true literally). Euler's method can be heuristically described in the following way:
  - The main idea is to think of moving forward in small time steps Δt.
  - At $t = t_0$, the slope is given by the $f(y_0, t_0)$.
  - In a time step of Δt, you would move a distance $f(y_0, t_0)\Delta t$.
  - Since you started at position y_0 at time t_0, your position would change to $y_0 + f(y_0, t_0)\Delta t$ at the new time $t_0 + \Delta t$. We write $y_1 = y_0 + f(y_0, t_0)\Delta t$ and $t_1 = t_0 + \Delta t$ as the new position and new time.
  - We now repeat the above steps, but our starting position and time are y_1 and t_1 rather than y_0 and t_0. So we get a new position and time which are $y_2 = y_1 + f(y_1, t_1)\Delta t$ and $t_2 = t_1 + \Delta t$.
  - In general, at step n, our position is given by $y_n = y_{n-1} + f(y_{n-1}, t_{n-1})\Delta t$.

Activities

Activity 7.3.2

Consider the initial value problem

$$\frac{dy}{dt} = 2t - 1, y(0) = 0.$$

(a) Use Euler's method with $\Delta t = 0.2$ to approximate the solution at $t_i = 0.2, 0.4, 0.6, 0.8$, and 1.0. Record your work in the following table and sketch the points (t_i, y_i) on the axes provided

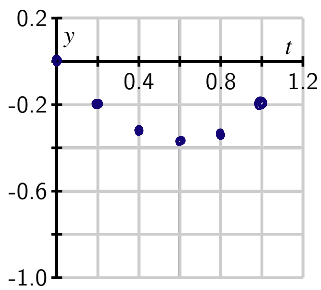

Figure 1: Grid for plotting points generated by Euler's method

t_i	y_i	dy/dt	Δy
0.0000	0.0000		
0.2000			
0.4000			
0.6000			
0.8000			
1.0000			

(b) Find the exact solution to the original initial value problem and use this function to find the error in your approximation at each one of the points t_i.

$\frac{dy}{dt} = 2t-1 \rightarrow \int \frac{dy}{dt} dt = \int 2t-1 dt \Rightarrow y = t^2 - t + C$ $\quad y(0) = 0 \Rightarrow y(0) = 1^2 - 1 + C$

$0 = C$

(c) Explain why the value y_5 generated by Euler's method for this initial value problem produces the same value as a left Riemann sum for the definite integral $\int_0^1 (2t - 1)\, dt$.

Writing the steps of Euler's method:

$y_5 = \Delta t f(t_0) + \Delta t f(t_1) + \Delta t f(t_2) + \Delta t f(t_3) + \Delta t f(t_4) \longrightarrow$ exactly the left Riemann sum of $f(t) = 2t-1$

(d) How would your computations differ if the initial value was $y(0) = 1$? What does this mean about different solutions to this differential equation?

They shift up by 1.

Activity 7.2.3

Consider the differential equation
$$\frac{dy}{dt} = 6y - y^2.$$

(a) Sketch the slope field for this differential equation on the axes provided in Figure 2

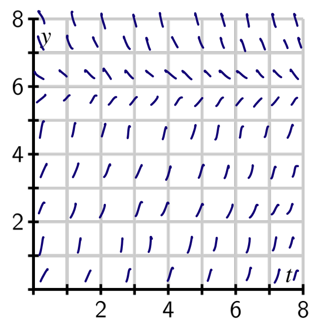

Figure 2: Grid for plotting the slope field

(b) Identify any equilibrium solutions and determine whether they are stable or unstable.

$\frac{dy}{dt} = 6y - y^2 = y(6-y) \Rightarrow y = 0$ or $y = 6$

From the slope field $y=0$ is unstable and $y=6$ is stable

(c) What is the long-term behavior of the solution that satisfies the initial value $y(0) = 1$?

The solution approaches $y=6$.

(d) Using the initial value $y(0) = 1$, use Euler's method with $\Delta t = 0.2$ to approximate the solution at $t_i = 0.2, 0.4, 0.6, 0.8,$ and 1.0. Record your results in the table below and sketch the corresponding points (t_i, y_i) on the axes provided on Figure 3. Note the different horizontal scale on the axes on Figure 2 and Figure 3

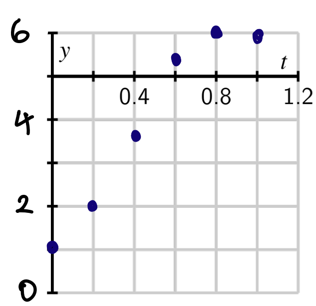

Figure 3: Grid for plotting points generated by Euler's method

t_i	y_i	dy/dt	Δy
0.0000	1.0000		
0.2000			
0.4000			
0.6000			
0.8000			
1.0000			

(e) What happens if we apply Euler's method to approximate the solution with $y(0) = 6$?

We should get $y_i = 6$ for all i.