§7.4 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 24 October - 28 October 2022

Separable Differential Equations

Main Concepts

- A **separable equation** is one that can be written in the following form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{f(t)}{g(y)}$$

- **Examples:**

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 t, \text{ where } f(t) = t \text{ and } g(y) = \frac{1}{y^2}$$

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{e^T x^3}{\sqrt{1+x^2}}, \text{ where } f(x) = \frac{x^3}{\sqrt{1+x^2}} \text{ and } g(T) = \frac{1}{e^T}$$

- **Not Examples:**

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 - t^2, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = e^{ux}, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y+x}$$

- **Solving** separable differential equations:
  - The key idea is to rewrite the equation as

$$g(y)\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

  - and integrate with respect to t:

$$\int g(y)\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = \int f(t)\,\mathrm{d}t.$$

  - Now we make the substitution $u = y$, thus $\mathrm{d}u = \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t$ and we get

$$\int g(u)\,\mathrm{d}u = \int f(t)\,\mathrm{d}t.$$

  - At this point, if both sides can be integrated, we end up with an equation involving u and t.
  - If you can solve u in terms of t, we're in good shape!

- <u>Don't</u> forget constants of integration or initial conditions (if they have been provided).

Activities

Activity 7.4.2

Suppose that the population of a town is growing continuously at an annual rate of 3% per year.

(a) Let $P(t)$ be the population of the town in year t. Write a differential equation that described the annual growth rate.

$$\frac{dP}{dt} = \frac{3}{100} P$$

(b) Find the solutions of this differential equation.

$$\int \frac{dP}{P} = \int \frac{3}{100}\,dt \Rightarrow \ln|P| = \frac{3}{100}t + C \qquad P = \underbrace{e^{C}}_{\text{call it } A} e^{3t/100}$$

$$\boxed{P = Ae^{3t/100}}$$

(c) If you know that the town's population in year 0 is 10.000, find the population $P(t)$.

$t=0$: $P = 10{,}000$

$10000 = Ae^{0} \qquad A = 10{,}000 \Rightarrow P(t) = 10{,}000e^{3t/100}$

(d) How long does it take for the population to double? This time is called the *doubling time*.

$P(t) = 20{,}000$ for some t. $\quad 20000 = 10000e^{3t/100}$

$2 = e^{3t/100} \qquad \ln 2 = 3t/100 \qquad \boxed{t = \frac{100\ln 2}{3}} \approx 23$ years

(e) Working more generally, find the doubling time if the annual growth rate is k times the population.

$\frac{dP}{dt} = kP \Rightarrow P = P_0 e^{kt}$ $\qquad 2\cancel{P_0} = \cancel{P_0}e^{kt}$

$P(0) = P_0$ $\qquad kt = \ln 2 \qquad \boxed{t = \frac{\ln 2}{k}}$

Activity 7.4.3

Suppose that a cup of coffee is initially at a temperature of 105° F and is placed in a 75° F room. If is the temperature of the coffee in degrees Fahrenheit at time t in minutes, Newton's law of cooling says that

$$\frac{dT}{dt} = -k(T - 75),$$

where k is a constant of proportionality.

(a) Suppose you measure that the coffee is cooling at one degree per minute at the time the coffee is brought into the room. Use the differential equation to determine the value of the constant k.

at $t=0$: $\frac{dT}{dt} = -1$ so $-1 = -k(105-75)$

$k = 1/30$

(b) Find all the solutions of this differential equation.

$\frac{dT}{dt} = -\frac{1}{30}(T-75)$ $\ln|T-75| = -\frac{t}{30} + C$ | $T = 75 + Ae^{-t/30}$ $t=0 \to T=105$ | $105 = 75 + Ae^{0}$ $30 = A$ $\Rightarrow T = 75 + 30e^{-t/30}$

(c) What happens to all the solutions as $t \to \infty$? Explain how this agrees with your intuition.

$\lim_{t\to\infty} T(t) = 75°F$

(d) What is the temperature of the coffee after 20 minutes?

$T(20) = 75 + 30e^{\frac{-20}{30}} = 75 + 30e^{-2/3}$

(e) How long does it take for the coffee to cool to 80°?

$80 = 75 + 30e^{-t/30} \to \frac{5}{30} = e^{-t/30} \longrightarrow \frac{1}{6} = e^{-t/30}$

$\ln 6 = t/30$ $t = 30\ln 6$

Activity 7.4.4

Solve each of the following differential equations or initial value problems.

(a) $\frac{dy}{dt} - (2-t)y = 2 - t$

$\frac{dy}{dt} = (2-t)(y+1)$ $\int \frac{dy}{y+1} = \int (2-t)dt$

$\ln|y+1| = 2t - \frac{t^2}{2} + C$ $y = Ae^{2t-\frac{t^2}{2}} - 1$

(b) $\frac{1}{t}\frac{dy}{dt} = e^{t^2-2y}$

$\frac{dy}{dt} = te^{t^2-2y}$

$= te^{t^2}e^{-2y}$

| $\int e^{2y}dy = \int te^{t^2}$

$\frac{1}{2}e^{2y} = \int te^{t^2}$

| $u = t^2$

$du = 2t$

$e^{2y} = \int e^{u}du$

$e^{2y} = e^{u} + C$

| $e^{2y} = e^{t^2} + C$

$2y = \ln|e^{t^2} + C|$

$y = \frac{1}{2}\ln|e^{t^2} + C|$

(c) $y' = 2y + 2, y(0) = 2$

$$A = e^{2C}$$

$$\frac{dy}{dt} = 2y+2$$

$$\frac{dy}{2y+2} = dt$$

$$\frac{1}{2}\ln|2y+2| = t + C$$

$$\ln|2y+2| = 2t + 2C$$

$$2y+2 = Ae^{2t}$$

$$y = \frac{Ae^{2t}-2}{2}$$

$$y(0) = 2$$

$$2 = \frac{A-2}{2}$$

$$4 = A-2$$

$$A = 6$$

$$y(t) = \frac{6e^{2t}-2}{2} = 3e^{2t}-1$$

(d) $y' = 2y^2, y(-1) = 2$

$$\frac{dy}{dt} = 2y^2$$

$$\int \frac{dy}{y^2} = \int 2\,dt$$

$$-\frac{1}{y} = 2t + C$$

$$y = \frac{-1}{2t+C}$$

$$y(-1) = 2$$

$$\hookrightarrow\ 2 = \frac{-1}{-2+C}$$

$$-4 + 2C = -1$$

$$2C = 3$$

$$C = 3/2$$

$$y = \frac{-1}{\frac{3}{2}+2t}$$

(e) $\dfrac{dy}{dt} = \dfrac{-2ty}{t^2+1}, y(0) = 4.$

$$\int \frac{dy}{y} = \int \frac{-2t}{t^2+1}\,dt$$

u sub t^2+1

$$\ln|y| = -\int \frac{du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|t^2+1| + C$$

$$e^C$$

$$y = Ae^{-\ln|t^2+1|}$$

$$= \frac{A}{t^2+1}$$

$$y(0) = 4$$

$$\frac{A}{1} = 4 \qquad A = 4$$

$$y(t) = \frac{4}{t^2+1}$$