§7.5 **Name:** ______________________________

Fall MATH 1120 Lec003 24 October - 28 October 2022

Modelling with Differential Equations

Main Concepts

- Differential Equations naturally appear when modelling many real world problems.
- A general principle to remember is

$$\text{total rate of change} = \text{rate in} - \text{rate out}.$$

- The techniques we learnt in earlier sections, Euler's method, separable equations, slope fields etc. are useful in studying the equations obtained from modelling these systems.

Activities

Activity 7.5.2

Suppose that you have a bank account that grows by 0.17% every year. Let $A(t)$ be the amount of money in the account in year t.

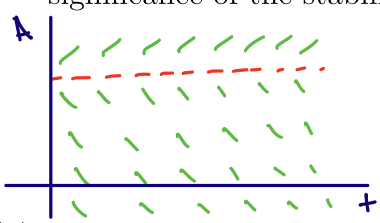
(a) What is the rate of change of A with respect to t?

$$\frac{dA}{dt} = \frac{0.17}{100} A$$

(b) Suppose that you are also withdrawing \$ 10,000 per year. Write a differential equation that expresses the total rate of change of A.

$$\frac{dA}{dt} = \frac{0.17}{100} A - 10000$$

(c) Sketch a slope field for this differential equation, find any equilibrium solutions, and identify them as either stable or unstable. Write a sentence or two that describes the significance of the stability of the equilibrium solution.



Equilibrium

$$\frac{0.17}{100} A = 10000$$

$$A = \frac{10^6}{0.17}$$

The equilibrium is unstable

(d) Suppose that you initially deposit \$ 100,000 into the account. How long does it take for you to deplete the account?

$$\frac{dA}{dt} = \frac{0.17}{100}\left(A - \frac{10^6}{0.17}\right)$$

$$\int \frac{dA}{A - \frac{10^8}{17}} = \int \frac{17}{10^4}\, dt$$

$$\ln\left|A - \frac{10^8}{17}\right| = \frac{17}{10^4} t + C$$

$$A = \frac{10^8}{17} + B e^{\frac{17}{10^4} t}$$

$A(0) = 10^5$

$$10^5 = \frac{10^8}{17} + B$$

$$B = 10^5 - 10^8/17$$

$A(t) = 0$

$$\Rightarrow \left(10^5 - \frac{10^8}{17}\right) e^{\frac{17}{10^4} t} = -\frac{10^8}{17}$$

$t \approx 10.08$ years

(e) What is the smallest amount of money you would need to have in the account to guarantee that you never deplete the money in the account?

This is the value of the equilibrium solution

$\approx \$5.9$ million

(f) If your initial deposit is \$ 30,000, how much could you withdraw every year without depleting the account?

$$0 = \frac{dA}{dt} = \frac{0.17}{100} \times 30{,}000 - W \qquad W = \$51$$

Activity 7.5.3

A dose of morphine is absorbed from the bloodstream of a patient at a rate proportional to the amount in the bloodstream.

(a) Write a differential equation for $M(t)$, the amount of morphine in the patient's bloodstream, using k as the constant of proportionality.

$\frac{dM}{dt} = -kM$ where $k > 0$

(b) Assuming that the initial dose of morphine is M_0, solve the initial value problem to find $M(t)$. Use the fact that the half-life for the absorption of morphine is two hours to find the constant k.

$M = M_0 e^{-kt}$ $\quad \frac{1}{2} = e^{-k2}$ $\quad \ln 2 = 2k$ $\quad k = \frac{\ln 2}{2}$

(c) Suppose that a patient is given morphine intravenously at the rate of 3 milligrams per hour. Write a differential equation that combines the intravenous administration of morphine with the body's natural absorption.

$\frac{dM}{dt} = 3 - kM$

(d) Find any equilibrium solutions and determine their stability.

$M = 3/k$

(e) Assuming that there is initially no morphine in the patient's bloodstream, solve the initial value problem to determine $M(t)$. What happens to $M(t)$ after a very long time?

$\int \frac{dM}{3-kM} = \int dt \Rightarrow -\frac{1}{k}\ln|3-kM| = t + C$

$M = \frac{3}{k}(1-e^{-kt})$

(f) To what rate should a doctor reduce the intravenous rate so that there is eventually 7 milligrams of morphine in the patient's bloodstream?

Suppose the rate is r. Then

$\frac{dM}{dt} = r - kM$ equilibrium $\rightarrow \frac{dM}{dt} = 0$ $\quad r = kM$ ($M \rightarrow$ 7mg)

$r = \frac{\ln 2}{2} 7 = 2.426$ mg/hr