POPULATION GROWTH AND THE LOGISTIC EQUATION

MAIN CONCEPTS

- The **Logistic equation** is used to describe population growth. The general form is given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(N - P).$$

- The number k is called the *per capita growth rate* and N is called the *carrying capacity*.
- The equilibria of this equation are $P = 0$ (unstable) and $P = N$ (stable).
- We usually consider the initial value problem where the population starts with P_0 individuals at $t = 0$. The solution to this initial value problem is given by

$$P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt} + 1}$$

Activities

We will study the Earth's population. To get started in the table below are some data for the earth's population in recent years that we will use in our investigations.

Year	1998	1999	2000	2001	2002	2005	2006	2007	2008	2009	2010
Pop (billions)	5.932	6.008	6.084	6.159	6.234	6.456	6.531	6.606	6.681	6.756	6.831

Activity 7.6.2

Our first model will be based on the following assumption:

The rate of change of the population is proportional to the population.

On the face of it, this seems pretty reasonable. When there is a relatively small number of people, there will be fewer births and deaths so the rate of change will be small. When there is a larger number of people, there will be more births and deaths so we expect a larger rate of change.

If $P(t)$ is the population t years after the year 2000, we may express this assumption as

$$\frac{dP}{dt} = kP,$$

where k is a constant of proportionality.

(a) Use the data in the table to estimate the derivative $P'(0)$ using a central difference

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

Assume that $t = 0$ corresponds to the year 2000.

$$P'(0) \approx \frac{6.159 - 6.008}{2} = 0.0755$$

(b) What is the population $P(0)$?

$P(0) = 6.084$ billion

(c) Use your results from (a) and (b) to estimate the constant of proportionality k in the differential equation.

$P' = kP$ so $k \approx 0.012$

(d) Now that we know the value of k, we have the initial value problem

$$\frac{dP}{dt} = kP, P(0) = 6.084.$$

Find the solution to this initial value problem.

$P(t) = 6.084\, e^{0.012t}$ billions

(e) What does your solution predict for the population in the year 2010? Is this close to the actual population given in the table?

$P(10) = 6.86$ billions

(f) When does your solution predict that the population will reach 12 billion?

$\frac{12}{6.084} = e^{0.012t}$ $\quad t = \frac{1}{0.012} \ln\left(\frac{12}{6.084}\right) \approx 56$ years

(g) What does your solution predict for the population in the year 2500?

2454.46 billion or 2.5 trillion

(h) Do you think this is a reasonable model for the earth's population? Why or why not? Explain your thinking using a couple of complete sentences.

No, it's not. We have finite resources, so the population must stagnate.

Activity 7.6.3

Consider the logistic equation

$$\frac{dP}{dt} = kP(N-P)$$

with the graph of $\frac{dP}{dt}$ vs. P shown on Figure 1

$N = 12.5$

$P_0 = 6.084$

$k = 0.002$

textbook:

$$P = \frac{12.5}{1.0546e^{-0.025t} + 1}$$

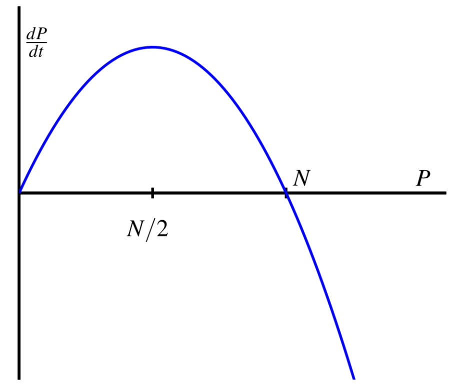

Figure 1: Plot of $\frac{dP}{dt}$ vs. P.

(a) At what value of P is the rate of change greatest?

$$P = \frac{N}{2}$$

(b) Consider the model for the earth's population that we created. At what value of P is the rate of change greatest? How does that compare to the population in recent years?

$$P = \frac{12.5}{2} = 6.125 \text{ billion}$$

(c) According to the model we developed, what will the population be in the year 2100?

$$P = \frac{12.5}{1.0546e^{-0.025t}+1}$$

$P(100) \approx 11.5$ billion

(d) According to the model we developed, when will the population reach 9 billion?

$$9 = \frac{12.5}{1.0546e^{-0.025t}+1}$$

$$1.0546e^{-0.025t} = \frac{12.5}{9} - 1 = 0.389$$

$$e^{-0.025t} = 0.369 \qquad t = \frac{\ln(0.369)}{-0.025} \approx 40 \text{ years}$$

(e) Now consider the general solution to the general logistic initial value problem that we found, given by

$$P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt}+1}.$$

Verify algebraically that $P(0) = P_0$ and that $\lim_{t\to\infty} P(t) = N$.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt}+1}$$

$$= \frac{N}{0+1} = N$$

since $e^{-kNt} \to 0$ as $t \to \infty$