

SEQUENCES

MAIN CONCEPTS

- A **sequence** is a (possibly infinite) list of numbers s_1, s_2, s_3, \dots in specified order. Examples:

(a) $1, 2, 3, 4, \dots$

(b) $2, 4, 6, 8, \dots$

(c) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

- s_n is referred to as the n^{th} term of the sequence.
- A sequence can be thought of as a function whose domain is the set of positive integers,

$$f(n) = s_n.$$

For example:

(a) $f(n) = n$ gives the sequence $1, 2, 3, 4, \dots$

(b) $f(n) = 2n$ gives the sequence $2, 4, 6, 8, \dots$

(c) $f(n) = \frac{1}{n}$ gives the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

- When we say that a sequence $\{a_n\}$ **converges** to the number L , we mean that a_n can be made as close to L as we like by taking n to be large enough.
- **Formal definition:** We say that $\{a_n\}$ **converges to the limit** L and write

$$\lim_{n \rightarrow \infty} a_n = L$$

if, for every $\epsilon > 0$, there exists $N > 0$ such that

$$|L - a_n| < \epsilon$$

for all $n > N$.

ACTIVITIES

ACTIVITY 1

Figure 1 shows triangular numbers. The n -th **triangular number** T_n is the number of dots arranged in an equilateral triangle with n dots on the side.

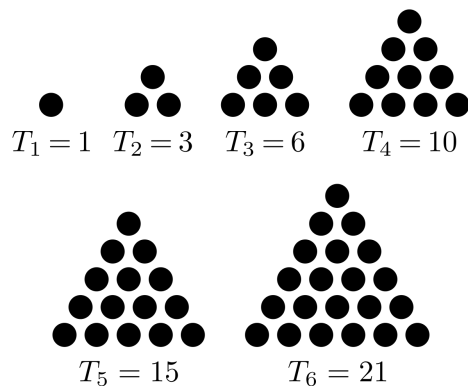


Figure 1: Triangular numbers

- (a) How is T_n related to T_{n-1} ? (**Hint:** try to argue using a picture with the triangles)

$$T_n = T_{n-1} + n$$

- (b) Using the result in the previous part, find a formula involving summation for T_n . Check that your formula works for $n = 1, 2, 3$.

$$T_n = \sum_{k=1}^n k$$

- (c) Find a formula for T_n that does not involve a summation sign. Check that your formula works for $n = 1, 2, 3$.

$$T_n = \frac{n(n+1)}{2}$$

ACTIVITY 2

Consider the sequence $\{T_n\}$ of triangular numbers.

- (a) What is the function $f(n)$ that defines the n -th term of the sequence $\{T_n\}$?

$$f(n) = \frac{n(n+1)}{2}$$

- (b) For each of the following expressions, identify if it is a number, sequence, or function:

(i) T_n number

(ii) $\{T_{n+1}\}$ sequence

(iii) T_3 number

(iv) $\{f(n)\}$ sequence

(v) $f(n)$ function or number

(vi) $f(3)$ number

ACTIVITY 3

In this activity we will try to build some intuition about the definition of the limit of a sequence.

- (a) Figure 2 shows a visual interpretation of the limit definition. We plot the sequence by

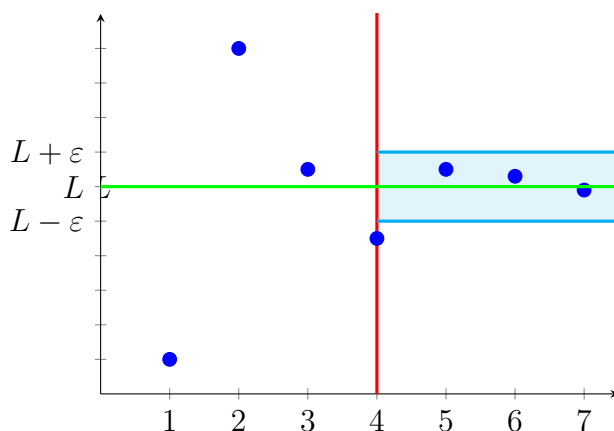


Figure 2: A visual interpretation of the limit

considering the graph $y = f(x)$ evaluated at integer values of x (so the plot consists of a bunch of disconnected points). We call the light blue region an “ ε -band around L ”, that is, the set of points in the plane whose y -coordinate lies between $L - \varepsilon$ and $L + \varepsilon$. Then the definition of limit could be restated as follows:

$\lim_{n \rightarrow \infty} a_n = L$ means that no matter how small ε is, we can always find a number N such that all points of the plotted sequence strictly to the right of the line $x = N$ lie within an ε -band around L .

- (b) For each of these parts of the original definition, find the corresponding concept on the visual one:

- (i) ... for every $\varepsilon > 0$...

Blue lines / shaded region (horizontal)

- (ii) ... there exists $N > 0$...

red vertical line

- (iii) ... $|L - a_n| < \varepsilon$ for all $n > N$.

blue dots falling inside the shaded region

(c) For this part, we'll play a game. The sequence $\{a_n\}$ is given. Player A will try to show that the limit of the sequence is L , and player B will try to disprove it.

- then player A chooses a number L
- then player B chooses a margin of error ε
- then player A chooses a number N
- then to win, player B has to find a number n larger than N such that a_n is further from L than the margin of error ε . If player B can't find such a number, player A wins.

Then the definition of a limit could be restated as follows:

If player A has a winning strategy after choosing L then the sequence converges to L .

(d) For each of these parts of the original definition, find the corresponding concept on the game one:

(i) ... for every $\varepsilon > 0$...

Player B's margin of error

(ii) ... there exists $N > 0$...

Player A's response

(iii) ... $|L - a_n| < \varepsilon$ for all $n > N$.

Player A wins the round