§8.1 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 31 October - 04 November 2022

Sequences contd.

Review

- **Formal definition:** We say that $\{a_n\}$ **converges to the limit** L and write

$$\lim_{n\to\infty} a_n = L$$

if, for every $\epsilon > 0$, there exists $N > 0$ such that

$$|L - a_n| < \epsilon$$

for all $n > N$.

Main Concepts

- **Theorem 1** If $\lim_{x\to\infty} f(x)$ exists, then the sequence $a_n = f(n)$ converges to the same limit
$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

- **Theorem 2** If f is continuous and $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

In other words, we may pass a limit of a sequence inside a continuous function.

Activities

Activity 1

Let $a_n = \frac{1+2n}{3n-2}$.

(a) Can you guess what $\lim_{n\to\infty} a_n$ is? (**Hint:** try to plug in a very large number for n)

$$a_n = \frac{\frac{1}{n} + 2}{3 - \frac{2}{n}} \longrightarrow \frac{2}{3}$$

(b) Compute $\left|a_n - \frac{2}{3}\right|$ and simplify it as much as possible.

$$\left| \frac{1+2n}{3n-2} - \frac{2}{3} \right| = \left| \frac{3+6n-6n+4}{9n-6} \right| = \left| \frac{7}{9n-6} \right|$$

(c) Consider your expression for $\left|a_n - \frac{2}{3}\right|$ from the previous part. Can you make it small by increasing n?

yes

(d) How large would n have to be to ensure that $\left|a_n - \frac{2}{3}\right| < \varepsilon$?

$$\frac{7}{9n-6} < \varepsilon \qquad \frac{7}{\varepsilon} < 9n-6 \qquad n > \frac{7}{9\varepsilon} + \frac{6}{9}$$

(e) Explain how this proves that $\lim_{n\to\infty} a_n = \frac{2}{3}$, with all three interpretations (the definition, the visual one, and the game one).

For every ε, we have found

$N = \left\lceil \frac{7}{9\varepsilon} + 6/9 \right\rceil \longrightarrow \lceil a \rceil$ is notation for rounding up a.

Activity 2

For each of the following sequences, find a function defining them and use Theorem 1 to find their limits.

(a) Let $\{a_n\}$ be the sequence starting $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$.

$$a_n = \frac{n+1}{n} \qquad f(x) = \frac{x+1}{x} \qquad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+1/x}{1}$$

$$= 1$$

(b) Let $b_n = \frac{n+\ln(n)}{n^2}$ (**Hint:** you might want to use L'Hôpital's rule)

$$f(x) = \frac{x+\ln(x)}{x^2} \qquad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+1/x}{2x}$$ L'Hopital

$$= 0$$

Activity 3

Let $f(x) = \sin(x\pi)$ and let $c_n = f(n)$.

(a) What is $\lim_{x\to\infty} f(x)$?

It doesn't exist (sin(πx) just keeps oscillating and doesn't converge)

(b) What is $\lim_{n\to\infty} c_n$?

$$c_n = \sin(n\pi) = 0$$

$$\lim_{n\to\infty} c_n = 0$$

(c) Does this contradict Theorem 1? Explain.

No since $\lim_{x\to\infty} f(x)$ doesn't exist.

Activity 4

In this activity we will compute $\lim_{n\to\infty} a_n$, where $\{a_n\}$ is the sequence

$$1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots$$

(a) Find a function $f(x)$ such that $f(n) = a_n$.

$a_n = n^{1/n}$ $\qquad f(x) = x^{1/x}$

L'Hopital

(b) Let $g(x) = \ln(f(x))$. Find

$$\lim_{x\to\infty} g(x)$$

$g(x) = \frac{1}{x} \ln x$ $\qquad \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1}$

$= 0$

(c) Consider the following argument

We found that $\lim_{x\to\infty} g(x) = 0$. *We also know that* $g(x) = \ln(f(x))$, *and* $\ln(x)$ *is continuous so therefore we have*

$$0 = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \ln(f(x)) = \ln\left(\lim_{x\to\infty} f(x)\right)$$

therefore, since $\ln(1) = 0$, *by Theorem 2, we can conclude that* $\lim_{x\to\infty} f(x) = 1$.

Is this argument correctly applying Theorem 2?

No. Theorem 2 says that if a_n converges to L then for a continuous function f, $\lim_{n\to\infty} f(a_n) = f(L)$.
That is not what we are doing in this problem

(d) Can you fix the argument in the previous part so that it correctly uses Theorem 2?

let $b_n = g(n) = \frac{\ln n}{n}$. e^x is a continuous function

Since $\lim_{n\to\infty} b_n = 0$ $\quad \lim_{n\to\infty} e^{b_n} = e^0 = 1$

$e^{b_n} = e^{\frac{\ln n}{n}} = e^{\ln (n)^{1/n}} = n^{1/n}$ Therefore $\lim_{n\to\infty} n^{1/n} = 1$

$\square$