§8.3 **Name:** Solutions
Fall MATH 1120 Lec003 14 November - 18 November 2022

Series of Real Numbers

Main Concepts

- An **infinite series** is the sum of all (infinitely many) terms in a sequence $\{a_n\}$,

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + ... + a_n + ...$$

- Examples:
  - $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...$
  - $\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k-1} = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$
  - $\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + ...$

- The n^{th} **partial sum** of the series $\sum_{k=1}^{\infty} a_k$ is the finite sum $S_n = \sum_{k=1}^{n} a_k$.

- The partial sums themselves can be thought of as a sequence $S_1, S_2, S_3, ..., S_n, ...$

- The infinite series $\sum_{k=1}^{\infty} a_k$ is said to **converge** if the sequence of partial sums $\{S_n\}$ converges. If $\lim_{n\to\infty} S_n = S$, then $\sum_{k=1}^{\infty} a_k = S$.

- The infinite series $\sum_{k=1}^{\infty} a_k$ is said to **diverge** if the $\{S_n\}$ does not converge.

- **The Divergence Test:** If $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum_{k=1}^{\infty} a_k$ diverges. This also means that if $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to} a_k = 0$.

  The divergence test only tells you whether a series diverges, not whether it converges.

- **The Integral Test:** The idea here is to compare the series to an improper integral and deduce convergence/divergence based on the convergence/divergence of the improper integral.

  Let f be a real-valued function and assume that f is decreasing and positive for all x larger than some number c. Let $a_k = f(k)$ for each positive integer k.

  a. If the improper integral $\int_c^{\infty} f(x)\,\mathrm{d}x$ converges, then series $\sum_{k=1}^{\infty} a_k$ converges.
  b. If the improper integral $\int_c^{\infty} f(x)\,\mathrm{d}x$ diverges , then series $\sum_{k=1}^{\infty} a_k$ diverges.

Activities

Activity 8.3.2

Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

While it is physically impossible to add an infinite collection of numbers, we can, of course, add any finite collection of them. In what follows, we investigate how understanding how to find the n-th partial sum (that is, the sum of the first n terms) enables us to make sense of the infinite sum.

(a) Sum the first two numbers in this series. That is, find a numeric value for

$$\sum_{k=1}^{2} \frac{1}{k^2}$$

(b) Next, add the first three numbers in the series.

(c) Continue adding terms in this series to complete the list below. Carry each sum to at least 8 decimal places.

$\sum_{k=1}^{1} \frac{1}{k^2} = 1$	$\sum_{k=1}^{2} \frac{1}{k^2} =$
$\sum_{k=1}^{3} \frac{1}{k^2} =$	$\sum_{k=1}^{4} \frac{1}{k^2} =$
$\sum_{k=1}^{5} \frac{1}{k^2} =$	$\sum_{k=1}^{6} \frac{1}{k^2} =$
$\sum_{k=1}^{7} \frac{1}{k^2} =$	$\sum_{k=1}^{8} \frac{1}{k^2} =$
$\sum_{k=1}^{9} \frac{1}{k^2} =$	$\sum_{k=1}^{10} \frac{1}{k^2} =$

(d) The sums above form a sequence whose n-th term is $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Based on your calculations above, do you think the sequence $\{S_n\}$ converges or diverges? Explain. How do you think this sequence $\{S_n\}$ is related to the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Activity 8.3.3

If the **series** $\sum a_k$ converges, then an important result necessarily follows regarding the **sequence** $\{a_n\}$. This activity explores this result. Assume that the series $\sum_{k=1}^{\infty} a_k$ converges and has sum equal to L.

(a) What is the n-th partial sum S_n of the series $\sum_{k=1}^{\infty} a_k$?

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

(b) What is the $(n-1)$-th partial sum S_{n-1} of the series $\sum_{k=1}^{\infty} a_k$?

$$S_{n-1} = \sum_{k=1}^{n-1} a_k = a_1 + a_2 + \cdots + a_{n-1}$$

(c) What is the difference between the n-th partial sum and the $(n-1)$-st partial sum of the series $\sum_{k=1}^{\infty} a_k$?

$$S_n - S_{n-1} = (a_1 + a_2 + \cdots + a_{n-1} + a_n) - (a_1 + a_2 + \cdots + a_{n-1})$$

$$= a_n$$

(d) Since we are assuming that $\sum_{k=1}^{\infty} a_k = L$, what does this tell us about $\lim_{n\to\infty} S_n$? Why? What does this tell us about $\lim_{n\to\infty} S_{n-1}$? Why?

$$\lim_{n\to\infty} S_n = L$$

$$\lim_{n\to\infty} S_{n-1} = L$$

(e) Combine the results of the previous parts of this activity to determine $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$.

$$\lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} a_n$$

$$0 = \lim_{n\to\infty} a_n$$

ACTIVITY 8.3.4 (OPTIONAL IN CLASS)

Determine if the Divergence Test applies to the following series. If the test does not apply, explain why. If the test does apply, what does it tell us about the series?

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k+1}$ $\displaystyle\lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0 \Rightarrow$ Diverges

(b) $\displaystyle\sum_{k=1}^{\infty} (-1)^k$ $\displaystyle\lim_{k\to\infty} (-1)^k$ doesn't exist $\Rightarrow$ Diverges

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ $\displaystyle\lim_{k\to\infty} \frac{1}{k} = 0 \Rightarrow$ Inconclusive

ACTIVITY 8.3.5

Consider the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Recall that the harmonic series will converge provided that its sequence of partial sums converges. The n-th partial sum S_n of the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ &= 1(1) + 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right) + \cdots + 1\left(\frac{1}{n}\right). \end{aligned}$$

Through this last expression for S_n, we can visualize this partial sum as a sum of areas of rectangles with heights $\frac{1}{m}$ and bases of length 1, as shown in Figure 1 The graph of the

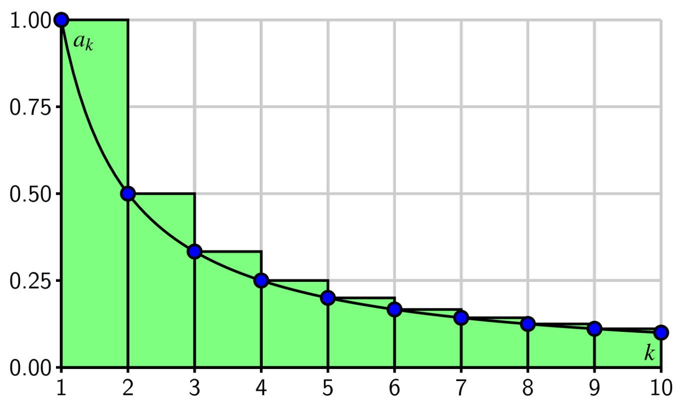

Figure 1: A picture of the 9th partial sum of the harmonic series as a sum of areas of rectangles

continuous function f defined by $f(x) = \frac{1}{x}$ is overlaid on this plot.

(a) Explain how this picture represents a particular Riemann sum.

It represents a left-handed Riemann sum. of the fuction $1/x$ between 1 and 10.

(b) What is the definite integral that corresponds to the Riemann sum you considered in (a)?

$$\int_1^{10} \frac{1}{x}\,dx$$

(c) Which is larger, the definite integral in (b), or the corresponding partial sum S_9 of the series? Why?

The Riemann sum is larger since $1/x$ is decreasing.

(d) If instead of considering the 9-th partial sum, we consider the n-th partial sum, and we let n go to infinity, we can then compare the series $\sum_{k=1}^{\infty} \frac{1}{k}$ to the improper integral $\int_1^{\infty} \frac{1}{x}\,dx$. Which of these quantities is larger? Why?

$$\int_1^{n} \frac{1}{x}\,dx < \sum_{k=1}^{n} \frac{1}{k}$$

(e) Does the improper integral $\int_1^{\infty} \frac{1}{x}\,dx$ converge or diverge? What does that result, together with your work in (d), tell us about the series $\sum_{k=1}^{\infty} \frac{1}{k}$?

$$\lim_{n\to\infty} \int_1^{n} \frac{1}{x}\,dx \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}$$

$$= \int_1^{\infty} \frac{1}{x}\,dx \leq \sum_{k=1}^{\infty} 1/k \implies \sum 1/k \text{ diverges}$$

Activity 8.3.6 (OPTIONAL IN CLASS)

The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ are special series called p-series. We have already seen that the p-series with $p = 1$ (the harmonic series) diverges. We investigate the behavior of other p-series in this activity.

(a) Evaluate the improper integral $\int_1^{\infty} \frac{1}{x^2}\,dx$. Does the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converge or diverge? Explain.

$$\int_1^{\infty} \frac{1}{x^2}dx = \lim_{n\to\infty} \int_1^{n} \frac{1}{x^2}dx = \lim_{n\to\infty} \left.\frac{-1}{x}\right|_1^n$$

$$= \lim_{n\to\infty} \frac{-1}{n} + 1 = 1 \qquad \sum_{k=1}^{\infty} 1/k^2 \text{ converges}$$

(b) Evaluate the improper integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ where $p > 1$. For which values of p can we conclude that the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges?

$$\int_1^{\infty} 1/x^p\,dx = \lim_{n\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^n = \frac{1}{p-1} \quad (p>1)$$

$$\sum_{k=1}^{\infty} 1/k^p \text{ converges as long as } p>1$$

(c) Evaluate the improper integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ where $p < 1$. What does this tell us about the corresponding p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$?

$$\int_1^{\infty} 1/x^p\,dx = \infty \text{ for } p<1 \text{ so } \sum_{k=1}^{\infty} 1/k^p$$

diverges for $p<1$

(d) Summarize your work in this activity by completing the following statement:

The p series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p>1$