§8.3 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 14 November - 18 November 2022

Series of Real Numbers (contd.)

Main Concepts

- **The Limit Comparison Test:** The idea here is to deduce convergence/divergence of a series by comparing it to a well-understood series.

  Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with positive terms. If

  $$\lim_{k\to\infty} \frac{b_k}{a_k} = c$$

  for some positive finite constant c, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converge or both diverge.

- **The Ratio Test:** The idea of this test is check if a given series is "approximately geometric". Since we understand when a geometric series converges/diverges, we may deduce properties of the given series.

  Let $\sum_{k=1}^{\infty} a_k$ be an infinite series. Suppose

  $$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = r$$

  a. If $0 \leq r < 1$, then the series converges.
  b. If $r > 1$, then the series diverges.
  c. If $r = 1$, then the test is inconclusive (it could covnerge or diverge).

Activities

Activity 8.3.7

Consider the series $\sum_{k=1}^{\infty} \frac{k+1}{k^3+2}$. Since the convergence and divergence of a series only depends on the behavior of the series for large values of k, we might examine the terms of this series more closely as k gets large.

(a) By computing the value of $\frac{k+1}{k^3+2}$ for $k = 100$ and $k = 1000$, explain why the terms $\frac{k+1}{k^3+2}$ are essentially $\frac{k}{k^3}$ when k is large.

$\frac{k+1}{k^3+2} \approx \frac{k}{k^3}$ for large k because k is much larger than 1 & 2.

(b) Let's formalize our observations in (a) a bit more. Let $a_k = \frac{k+1}{k^3+2}$ and $b_k = \frac{k}{k^3}$. Calculate

$$\lim_{k\to\infty} \frac{a_k}{b_k}.$$

What does the value of the limit tell you about a_k and b_k for large values of k? Compare your response from part (a).

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k+1}{k^3+2}\cdot\frac{k^3}{k} = \lim_{k\to\infty} \frac{\frac{k+1}{k}}{\frac{k^3+2}{k^3}}$$

$$= \lim_{k\to\infty} \frac{1+1/k}{1+2/k^3} = 1$$

(c) Does the series $\sum_{k=1}^{\infty} \frac{k}{k^3}$ converge or diverge? Why? What do you think that tells us about the convergence or divergence of the series $\sum_{k=1}^{\infty} \frac{k+1}{k^3+2}$? Explain.

$\sum_{k=1}^{\infty} \frac{k}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $\qquad \sum_{k=1}^{\infty} a_k \approx \sum_{k=1}^{\infty} b_k$

so $\sum_{k=1}^{\infty} a_k$ converges.

ACTIVITY 8.3.8

Use the limit comparison test to determine the convergence or divergence of the series

$$\sum_{k=1}^{\infty} \frac{3k^2+1}{5k^4+2k+2}$$

by comparing it to the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

$$a_k = \frac{3k^2+1}{5k^4+2k+2} \qquad b_k = 1/k^2 = \frac{k^2}{k^4}$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{3k^2+1}{5k^4+2k+2} \cdot \frac{k^4}{k^2}$$

$$= \lim_{k\to\infty} \frac{\frac{3k^2+1}{k^2}}{\frac{5k^4+2k+2}{k^4}} = \lim_{k\to\infty} \frac{3 + 1/k^2}{5 + 2/k^3 + 2/k^4}$$

$$= \frac{3}{5} \implies \sum a_k \approx \frac{3}{5} \sum b_k$$

so $\sum a_k$ converges.

ACTIVITY 8.3.9

Consider the series defined by

$$\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}$$

This series is not a geometric series, but this activity will illustrate how we might compare this series to a geometric one. Recall that a series $\sum_{k=1}^{\infty} a_k$ is geometric if the ratio $\frac{a_{k+1}}{a_k}$ is always the same. For the series given above, note that $a_k = \frac{2^k}{3^k - k}$.

(a) To see if $\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}$ is comparable to a geometric series, we analyse the ratios of successive terms in the series. Complete Table 1 listing your calculations to at least 8 decimal places.

k	5	10	20	21	22	23	24	25
$\frac{a_{k+1}}{a_k}$								

Table 1

the ratio is approximately $0.\overline{6}$

(b) Based on your calculations, what can we say about the ratio $\frac{a_{k+1}}{a_k}$ if k is large?

$\frac{a_{k+1}}{a_k} \approx 0.\overline{6}$ for large k

(c) Do you agree or disagree with the statement "the series $\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}$ is approximately geometric when k is large"? If not, why not? If so do you think the series $\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}$ converges or diverges? Explain.

Yes. Each consequent term is approx. 0.66 times the previous term.

Activity 8.3.10

Determine whether each of the following series converges or diverges. Explicitly state which test you use.

(a) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

$a_k = \frac{k}{2^k}$ $\quad \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$

$= \lim_{k\to\infty} \frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}} = \lim_{k\to\infty} \frac{k+1}{2k} = \frac{1}{2}$

$\Rightarrow \sum_{k=1}^{\infty} \frac{k}{2^k}$ converges by ratio test.

(b) $\sum_{k=1}^{\infty} \frac{k^3+2}{k^2+1}$

$a_k = \frac{k^3+2}{k^2+1}$ $\quad \lim_{k\to\infty} \frac{k^3+2}{k^2+1} = \infty$ so

the series diverges by the divergence test

(c) $\sum_{k=1}^{\infty} \frac{10^k}{k!}$

$a_k = \frac{10^k}{k!}$ $\quad \frac{a_{k+1}}{a_k} = \frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k} = \frac{10}{k+1}$

$\lim_{k\to\infty} \frac{10}{k+1} = 0$. Converges by ratio test

(d) $\sum_{k=1}^{\infty} \frac{k^3-2k^2+1}{k^6+4}$

$a_k = \frac{k^3-2k^2+1}{k^6+4}$ $\quad$ let $b_k = \frac{k^3}{k^6}$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1 - 2/k + 1/k^3}{1 + 4/k^6} = 1$. $\sum a_k$ converges since $\sum b_k$ does (limit comparison test)