§8.3 **Name:** ____________________

Fall MATH 1120 Lec003 14 November - 18 November 2022

Series of Real Numbers (contd.)

Main Concepts

- **The Limit Comparison Test:** The idea here is to deduce convergence/divergence of a series by comparing it to a well-understood series.

  Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with positive terms. If

  $$\lim_{k\to\infty} \frac{b_k}{a_k} = c$$

  for some positive finite constant c, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converge or both diverge.

- **The Ratio Test:** The idea of this test is check if a given series is "approximately geometric". Since we understand when a geometric series converges/diverges, we may deduce properties of the given series.

  Let $\sum_{k=1}^{\infty} a_k$ be an infinite series. Suppose

  $$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = r$$

  a. If $0 \leq r < 1$, then the series converges.

  b. If $r > 1$, then the series diverges.

  c. If $r = 1$, then the test is inconclusive (it could covnerge or diverge).

Activities

Activity 8.3.7

Consider the series $\sum_{k=1}^{\infty} \frac{k+1}{k^3+2}$. Since the convergence and divergence of a series only depends on the behavior of the series for large values of k, we might examine the terms of this series more closely as k gets large.

(a) By computing the value of $\frac{k+1}{k^3+2}$ for $k = 100$ and $k = 1000$, explain why the terms $\frac{k+1}{k^3+2}$ are essentially $\frac{k}{k^3}$ when k is large.

(b) Let's formalize our observations in (a) a bit more. Let $a_k = \frac{k+1}{k^3+2}$ and $b_k = \frac{k}{k^3}$. Calculate

$$\lim_{k\to\infty} \frac{a_k}{b_k}.$$

What does the value of the limit tell you about a_k and b_k for large values of k? Compare your response from part (a).

(c) Does the series $\sum_{k=1}^{\infty} \frac{k}{k^3}$ converge or diverge? Why? What do you think that tells us about the convergence or divergence of the series $\sum_{k=1}^{\infty} \frac{k+1}{k^3+2}$? Explain.

Activity 8.3.8

Use the limit comparison test to determine the convergence or divergence of the series

$$\sum_{k=1}^{\infty} \frac{3k^2+1}{5k^4+2k+2}$$

by comparing it to the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Activity 8.3.9

Consider the series defined by

$$\sum_{k=1}^{\infty} \frac{2^k}{3^k-k}$$

This series is not a geometric series, but this activity will illustrate how we might compare this series to a geometric one. Recall that a series $\sum_{k=1}^{\infty} a_k$ is geometric if the ratio $\frac{a_{k+1}}{a_k}$ is always the same. For the series given above, note that $a_k = \frac{2^k}{3^k-k}$.

(a) To see if $\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}$ is comparable to a geometric series, we analyse the ratios of successive terms in the series. Complete Table 1 listing your calculations to at least 8 decimal places.

k	5	10	20	21	22	23	24	25
$\frac{a_{k+1}}{a_k}$								

Table 1

(b) Based on your calculations, what can we say about the ratio $\frac{a_{k+1}}{a_k}$ if k is large?

(c) Do you agree or disagree with the statement "the series $\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}$ is approximately geometric when k is large"? If not, why not? If so do you think the series $\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}$ converges or diverges? Explain.

Activity 8.3.10

Determine whether each of the following series converges or diverges. Explicitly state which test you use.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3+2}{k^2+1}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k!}$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3-2k^2+1}{k^6+4}$