§8.4 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 21 November - 25 November 2022

Alternating Series

Main Concepts

- An **Alternating series** is an infinite series of the form

$$\sum_{k=0}^{\infty}(-1)^k a_k \quad \text{or} \quad \sum_{k=1}^{\infty}(-1)^{k+1} a_k$$

  where $a_k > 0$ for each k.

- Examples:
  - $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... +$
  - $\sum_{k=0}^{\infty}(-1)^k 2^k = 1 - 2 + 4 - 8 + 16 - ...$

- **Alternating Series test:** If $a_k > a_{k+1}$ for all k and $\lim_{k\to\infty} a_k = 0$, then the alternating series $\sum(-1)^k a_k$ converges. In other words, if the sequence $\{a_k\}$ is decreasing to zero as $k \to \infty$, then the alternating series converges.

- Estimating alternating series: If the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ has positive terms a_k that decrease to zero as $k \to \infty$ and $S_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ is the n^{th} partial sum of the alternating series, then

$$\left|\sum_{k=1}^{\infty}(-1)^{k+1}a_k - S_n\right| \leq a_{n+1}.$$

  In other words, the difference between the n^{th} partial sum and the infinite series cannot be larger than the $(n+1)^{\text{th}}$ term of the sequence $\{a_k\}$.

- Suppose now that $\sum_{k=1}^{\infty} a_k$ is an infinite series whose terms can be positive or negative. It is called **absolutely convergent** if $\sum |a_k|$ converges. In other words, if you remove all the negative signs, the infinite sum still converges.

  The series is called **conditionally convergent** if $\sum |a_k|$ diverges, *but* $\sum a_k$ converges. In other words, when some of the terms are negative the sum converges, but when all the signs are made positive, it diverges.

- Examples:
  - The series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + ...$ is absolutely convergent because when the negative signs are removed, $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...$ is still convergent.
  - The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$ is convergent but not absolutely convergent because when the negative signs are removed, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$ is not convergent. Thus this is an example of a conditionally convergent series.

Activities

Activity 8.4.2

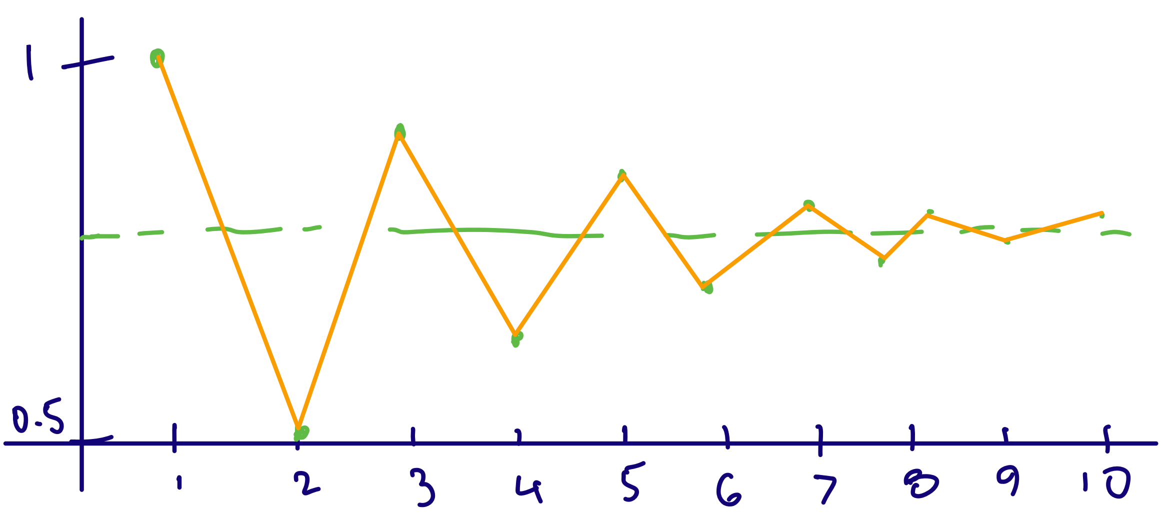
Remember that, by definition, a series converges if and only if its corresponding sequence of partial sums converges.

(a) Calculate the first few partial sums (to 10 decimal places) of the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}.$$

Label each partial sum with the notation $S_n = \sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}$ for an appropriate choice of n.

(b) Plot the sequence of partial sums from part (a). What do you notice about this sequence?



Activity 8.4.3

Which series converge and which diverge? Justify your answers.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+2}$

$$a_k = \frac{1}{k^2+2}$$

$$\lim_{k\to\infty} a_k = 0 \implies \text{Convergence}$$

(b) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}2k}{k+2}$

$$a_k = \frac{2k}{k+2}$$

$$\lim_{k\to\infty} a_k = \frac{2}{1+2/k} = 2 \implies \text{Divergence}$$

(c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(k)}$

$$a_k = \frac{1}{\ln k}$$

$$\lim_{n\to\infty} \frac{1}{\ln k} = 0 \implies \text{Convergence}$$

Activity 8.4.4

Determine the number of terms it takes to approximate the sum of the convergent alternating series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$$

$a_k = 1/k^4$

to within 0.0001.

$$\left| S_n - \sum \frac{(-1)^{k+1}}{k^4} \right| \leq \frac{1}{(n+1)^4} < 0.0001 = 10^{-4}$$

$(n+1)^4 = 10^4 \quad n \approx 10. \quad S_{10} = 0.94699$

Activity 8.4.5

(a) Explain why the series

$$1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \cdots$$

must have a sum that is less than the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$ is greater than the given series because all the operations are additions

(b) Explain why the series

$$1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \cdots$$

must have a sum that is greater than the series

$$\sum_{k=1}^{\infty} -\frac{1}{k^2}.$$

$\sum_{k=1}^{\infty} \frac{-1}{k^2}$ is lesser than the given series because all the terms are subtracted

(c) Given that the terms in the series

$$1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \cdots$$

converge to 0, what do you think the previous two results tell us about the convergence status of the series?

Since $\sum 1/k^2$ & $\sum -1/k^2$ both converge, we expect the given series to converge to a number between $\sum 1/k^2$ & $\sum -1/k^2$.

Activity 8.4.6

(a) Consider the series $\sum_{k=1}^{\infty}(-1)^k \frac{ln(k)}{k}$.

(i) Does this series converge? Explain

$a_k = \frac{\ln k}{k}$ $\lim_{k \to \infty} \frac{\ln k}{k} = 0$ (L'Hopital's rule)

So the series converges.

(ii) Does this series converge absolutely? Explain what test you use to determine your answer.

$\sum |a_k| = \sum \frac{\ln k}{k}$ diverges

Integral test. The seies doesn't converge absolutely

$\int_1^\infty \frac{\ln x}{x} = \lim_{n\to\infty} \int_1^n \frac{\ln x}{x} dx$ $u = \ln x$ $du = \frac{1}{x} dx$

$= \lim_{n\to\infty} \int_0^{\ln n} u\,du = \lim_{n\to\infty} \frac{u^2}{2}\Big|_0^{\ln n} = \lim_{n\to\infty} \frac{(\ln n)^2}{2} = \infty$

So by integral test, $\sum \frac{\ln k}{k}$ diverges

(b) Consider the series $\sum_{k=1}^{\infty}(-1)^k \frac{ln(k)}{k^2}$.

(i) Does this series converge? Explain

Yes. It converges since $a_k = \frac{\ln k}{k^2} \longrightarrow 0$

(ii) Does this series converge absolutely? Hint: use the fact that $\ln(k) < \sqrt{k}$ for large values of k and then compare to an appropriate p-series.

Yes. Because $\sum \frac{\ln k}{k^2}$ converges.

$\int_1^\infty \frac{\ln x}{x^2} dx = \int_1^\infty \frac{\ln x}{x} \frac{1}{x} dx = \int_0^\infty \frac{u}{e^u} du = 1$ (integrate-by-parts)

So it converges by the integral test.