§8.5 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 28 November - 02 December 2022

Taylor Polynomials and Taylor Series

Main Concepts

- Roughly speaking, the **Taylor Series** of a function at a given point is an infinite series whose terms depend on the derivatives of the function at that point.
- The n^{th} order **Taylor Polynomial** of a function f centred at the point a is given by

$$\begin{aligned} P_n(x) &= f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \end{aligned}$$

  Here, $f^{(k)}(a)$ denotes the k^{th} derivative of f at the point a, e.g. $f^{(0)}(a) = f(a)$, $f^{(1)}(a) = f'(a)$, $f^{(2)}(a) = f''(a)$, and so on.

- Properties of the Taylor Polynomial:
  - $P_n(x)$ is an n^{th} degree polynomial,
  - $P_n(x)$ approximates $f(x)$ near the point $x = a$,
  - The taylor polynomial and its derivatives match that of the function at the point $x = a$, i.e., $P_n^{(k)}(a) = f^{(k)}(a)$.
- We say a function is **infinitely differentiable** at a point a if all the derivatives of f exist at the point $x = a$.
- Let f be an infinitely differentiable function. The **Taylor series** for f centred at $x = a$ is the series $T_f(x)$ defined by

$$T_f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}\cdot(x-a)^k$$

  Notice that the Taylor Polynomial, $P_n(x)$ is the n^{th} partial sum of the Taylor series $T_f(x)$.

- Examples:
  (a) Taylor series of e^x at $x = 0$: $1 + x + \frac{x}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$
  (b) Taylor series of $\frac{1}{1-x}$ at $x = 0$: $1 + x + x^2 + x^3 + x^4 + \ldots$
  (c) Taylor series of $\ln(x)$ at $x = 1$: $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-3)^3 - \frac{1}{4}(x-4)^4 + \ldots$

Activities

Activity 8.5.2

We have just seen that the n-th order Taylor polynomial centered at $a = 0$ for the exponential function e^x is

$$\sum_{k=0}^{n} \frac{x^k}{k!}.$$

In this activity, we determine small order Taylor polynomials for several other familiar functions, and look for general patterns.

(a) Let $f(x) = \frac{1}{1-x}$.

(i) Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\frac{1}{1-x}$ centered at 0.

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$1/(1-x)$	1
1	$1/(1-x)^2$	1
2	$2/(1-x)^3$	2
3	$6/(1-x)^4$	6
4	$24/(1-x)^5$	24

$$P_4(x) = 1 + x + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \frac{24x^4}{4!}$$

$$= 1 + x + x^2 + x^3 + x^4$$

(ii) Based on your results from part (i), determine a general formula for $f^{(k)}(0)$.

$$f^{(k)}(0) = k!$$

(b) Let $f(x) = \cos(x)$.

(i) Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for ~~$\frac{1}{1-x}$~~ centered at 0.

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1

$$P_4(x) = 1 + \frac{0 \cdot x}{1!} + \frac{(-1)\, x^2}{2!} + \frac{0 \cdot x^3}{3!} + \frac{(1)\, x^4}{4!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

(ii) Based on your results from part (i), determine a general formula for $f^{(k)}(0)$.

$$f^{(k)}(0) = \begin{cases} 0 & k \text{ is odd} \\ (-1)^j & \text{if } k = 2j \end{cases}$$

→ this is another way of saying k is even

(c) Let $f(x) = \sin(x)$.

(i) Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\frac{1}{1-x}$ centered at 0.

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0

$$P_4(x) = x - \frac{x^3}{3!}$$

(ii) Based on your results from part (i), determine a general formula for $f^{(k)}(0)$.

$$f^{(k)}(0) = \begin{cases} 0 & k \text{ is even} \\ (-1)^j & k = 2j+1 \end{cases}$$

→ another way of saying k is odd

Activity 8.5.3

In the previous activity we determined small order Taylor polynomials for a few familiar functions, and also found general patterns in the derivatives evaluated at 0. Use that information to write the Taylor series centered at 0 for the following functions.

(a) $f(x) = \frac{1}{1-x}$

$$\sum_{k=0}^{\infty} x^k$$

(b) $f(x) = \cos(x)$ (you will need to carefully consider how to indicate that many of the coefficients are 0. Think about a general way to represent an even integer.)

$$\sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!}$$

(c) $f(x) = \sin(x)$ (you will need to carefully consider how to indicate that many of the coefficients are 0. Think about a general way to represent an odd integer.)

$$\sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!}$$

(d) $f(x) = \frac{1}{1+x}$.

$$\sum_{k=0}^{\infty} (-1)^k x^k$$

Activity 8.5.4

For this activity, it might be useful if you use this Desmos worksheet: https://www.desmos.com/calculator/ttxdm690xm

(a) Plot the graphs of several of the Taylor polynomials centered at 0 (of order at least 5) for e^x and convince yourself that these Taylor polynomials converge to e^x for every value of x.

We did these in class using the online app

(b) Draw the graphs of several of the Taylor polynomials centered at 0 (or order at least 6) for $\cos(x)$ and convince yourself that these Taylor polynomials converge to $\cos(x)$ for every value of x. Write the Taylor series centered at 0 for $\cos(x)$.

(c) Draw the graphs of several of the Taylor polynomials centered at 0 for $\frac{1}{1-x}$. Based on your graphs, for what values of x do these Taylor polynomials appear to converge to $\frac{1}{1-x}$? How is this different from what we observe with e^x and $\cos(x)$? In addition, write the Taylor series centered at 0 for $\frac{1}{1-x}$.