§8.5 **Name:** SOLUTIONS

Fall MATH 1120 Lec003 28 November - 02 December 2022

# Taylor Polynomials and Taylor Series (contd.)

## Main Concepts

- Recall that for an infinitely differentiable function $f$, we can define its Taylor series centred at a point $x = a$, $T_f(x)$.
- For each $x$, the Taylor series is an infinite series of real numbers and it may or may not converge.
- Consider a Taylor series of the form

$$\sum_{k=0}^{\infty} c_k \cdot (x-a)^k$$

  and denote the limit of the ratio:

$$\lim_{k\to\infty} \frac{|c_{k+1}|}{|c_k|} = L.$$

  - If $L = 0$: The series converges on the whole real line $(-\infty, \infty)$.
  - If $L = \infty$: The series converges only at the point $x = a$.
  - If $0 < L < \infty$: The Taylor series converges absolutely for all $x$ in the interval $\left(a - \frac{1}{L}, a + \frac{1}{L}\right)$.
- The interval $\left(a - \frac{1}{L}, a + \frac{1}{L}\right)$ is called the **interval of convergence** and $\frac{1}{L}$ is called the **radius of convergence**.
- The Taylor polynomial of a function $f$ centred at $a$ approximates $f$ at points close to $a$. The **Lagrange error bounds** tell us how good this approximation is at some point $c$ that is close to $a$:

  Suppose $f$ is continuous and has $n+1$ continuous derivatives. Suppose also that $\left|f^{(n+1)}\right|(x) \leq M$ on the interval $[a, c]$. If $P_n(x)$ is the $n^{\text{th}}$ degree Taylor polynomial of $f$ centred at $a$, then

$$|P_n(c) - f(c)| \leq M \frac{|c-a|^{n+1}}{(n+1)!}.$$

## Activities

### Activity 8.5.5

(a) Use the Ratio test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \frac{1}{1-x}$ centered at $x = 0$.

$T_f(x) = \sum_{k=0}^{\infty} x^k \qquad C_k = 1 \qquad \lim_{k\to\infty} \frac{|C_{k+1}|}{|C_k|} = \lim_{k\to\infty} \frac{1}{1} = 1$

interval of convergence: $(0-1, 0+1) = (-1, 1)$

(b) Use the Ratio test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \cos(x)$ centered at $x = 0$.

$T_f(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^j \qquad C_{2j} = \frac{(-1)^j}{(2j)!}$

The interval of convergence is $(-\infty, \infty)$

Since odd terms are zero we look at $\lim_{j\to\infty} \frac{|C_{2(j+1)}|}{|C_{2j}|} = \frac{(2j)!}{(2j+2)!} = \frac{1}{(2j+2)(2j+1)} \longrightarrow \underline{\underline{0}}$

(c) Use the Ratio test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \sin(x)$ centered at $x = 0$.

Same idea as (b), Interval of convergene is $(-\infty, \infty)$

### Activity 8.5.6

Let $P_n(x)$ be the $n$-th order Taylor polynomial for $\sin(x)$ centered at $x = 0$. Determine how large we need to choose $n$ such that $P_n(2)$ approximates $\sin(2)$ to 20 decimal places.

Since $\sin x$ & $\cos x$ are bounded by 1, we can take $M=1$. Using Lagrange's error bound, we have $|P_n(2) - \sin(2)| \leq M \frac{|2-0|^{n+1}}{(n+1)!} = \frac{2^{n+1}}{(n+1)!}$. We want the error to be smaller than $10^{-20}$ i.e. $\frac{2^{n+1}}{(n+1)!} \leq 10^{-20}$. Solving for $n$, we get $n \geqslant 27$.

Activity 8.5.7

(a) Show that the Taylor series centered at 0 for $\cos(x)$ converges to $\cos(x)$ for every real number $x$.

from divergence test

Using the Lagrange formula, $|P_n(x) - \cos x| \leq \frac{|x|^{n+1}}{(n+1)!}$. We know that $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges for all $x$, therefore $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

therefore as $n \to \infty$ $|P_n(x) - \cos x| \longrightarrow 0$.

(b) Next we consider the Taylor series for $e^x$.

(i) Show that the Taylor series centered at 0 for $e^x$ converges to $e^x$ for every nonnegative value of $x$.

Since $e^x$ is an increasing, on $[0, c]$, $e^x \leq e^c$. We take $M = e^c$. Then the Lagrange formula says that $|P_n(x) - e^x| \leq e^c \frac{|x|^{n+1}}{(n+1)!}$ and as $n \to \infty$ the right hand side goes to zero.

(ii) Show that the Taylor series centered at 0 for $e^x$ converges to $e^x$ for every ~~non~~negative value of $x$.

For negative $x$, $e^x \leq 1$. So $M = 1$, and apply the same idea from (ii)

(iii) Explain why the Taylor series centered at 0 for $e^x$ converges to $e^x$ for every real number $x$. Recall that we earlier showed that the Taylor series centered at 0 for $e^x$ converges for all $x$, and we have now completed the argument that the Taylor series for $e^x$ actually converges to $e^x$ for all $x$.

The Taylor series converges for every $x \geqslant 0$ from (i) & for $x < 0$ from (ii). So it converges for all $x$.

(c) Let $P_n(x)$ be the $n$-th order Taylor polynomial for $e^x$ centered at 0. Find a value of $n$ such that $P_n(5)$ approximates $e^5$ correct to 8 decimal places.

$M = e^5$ $\quad |P_n(5) - e^5| \leq \frac{e^5 \, 5^{n+1}}{(n+1)!} \leq 10^{-8}$

solving gives $n = 28$.