§8.5 **Name:** ____________________

Fall MATH 1120 Lec003 28 November - 02 December 2022

Taylor Polynomials and Taylor Series (contd.)

Main Concepts

- Recall that for an infinitely differentiable function f, we can define its Taylor series centred at a point $x = a$, $T_f(x)$.
- For each x, the Taylor series is an infinite series of real numbers and it may or may not converge.
- Consider a Taylor series of the form

$$\sum_{k=0}^{\infty} c_k \cdot (x-a)^k$$

and denote the limit of the ratio:

$$\lim_{k\to\infty} \frac{|c_{k+1}|}{|c_k|} = L.$$

  - If $L = 0$: The series converges on the whole real line $(-\infty, \infty)$.
  - If $L = \infty$: The series converges only at the point $x = a$.
  - If $0 < L < \infty$: The Taylor series converges absolutely for all x in the interval $\left(a - \frac{1}{L}, a + \frac{1}{L}\right)$.
- The interval $\left(a - \frac{1}{L}, a + \frac{1}{L}\right)$ is called the **interval of convergence** and $\frac{1}{L}$ is called the **radius of convergence**.
- The Taylor polynomial of a function f centred at a approximates f at points close to a. The **Lagrange error bounds** tell us how good this approximation is at some point c that is close to a:

  Suppose f is continuous and has $n+1$ continuous derivatives. Suppose also that $\left|f^{(n+1)}\right|(x) \leq M$ on the interval $[a, c]$. If $P_n(x)$ is the n^{th} degree Taylor polynomial of f centred at a, then

$$|P_n(c) - f(c)| \leq M \frac{|c-a|^{n+1}}{(n+1)!}.$$

Activities

Activity 8.5.5

(a) Use the Ratio test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \frac{1}{1-x}$ centered at $x = 0$.

(b) Use the Ratio test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \cos(x)$ centered at $x = 0$.

(c) Use the Ratio test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \sin(x)$ centered at $x = 0$.

Activity 8.5.6

Let $P_n(x)$ be the n-th order Taylor polynomial for $\sin(x)$ centered at $x = 0$. Determine how large we need to choose n such that $P_n(2)$ approximates $\sin(2)$ to 20 decimal places.

Activity 8.5.7

(a) Show that the Taylor series centered at 0 for $\cos(x)$ converges to $\cos(x)$ for every real number x.

(b) Next we consider the Taylor series for e^x.

(i) Show that the Taylor series centered at 0 for e^x converges to e^x for every nonnegative value of x.

(ii) Show that the Taylor series centered at 0 for e^x converges to e^x for every nonnegative value of x.

(iii) Explain why the Taylor series centered at 0 for e^x converges to e^x for every real number x. Recall that we earlier showed that the Taylor series centered at 0 for e^x converges for all x, and we have now completed the argument that the Taylor series for e^x actually converges to e^x for all x.

(c) Let $P_n(x)$ be the n-th order Taylor polynomial for e^x centered at 0. Find a value of n such that $P_n(5)$ approximates e^5 correct to 8 decimal places.