§2.3, 2.4 **Name:** ____________________

Spring MATH 2220 29 January 2020

1 Differentiation of Multivariate functions

1. Given a function $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, where U is open, we define the *partial derivative* of f with respect to x as:

$$f_x = \frac{\partial f}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h},$$

if it exists. The partial derivatives f_y and f_z are defined similarly.

2. The *linear approximation* to $f(x, y)$ at (x_0, y_0) is

$$\ell_{(x_0,y_0)}(x, y) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0).$$

3. If f is differentiable at (x_0, y_0), the *tangent plane* to the graph of f at (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$ is

$$z = \ell_{(x_0,y_0)}(x, y).$$

4. More generally, for functions $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, we define the $m \times n$ *derivative matrix* (also referred to as the *Jacobian*):

$$\mathrm{D}f(\mathbf{x_0}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix},$$

where all the partial derivatives are evaluated at $\mathbf{x}_0$.

5. We say that a function is *differentiable* at $\mathbf{x}_0$ if all the partials exist **and**

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathrm{D}f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

6. Given a function $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, its *gradient* is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

It is straightforward to extend this concept to higher dimensions.

2 Practice Problems

1. Compute f_x, f_y and f_z for the following functions:

   (a) $f(x, y, z) = x^2 - y^2 + 2z^2$

   (b) $f(x, y, z) = \log(x^2 + 2y^2 - 3z^2)$

   (c) $f(x, y, z) = x^{y^z}$

2. Find the equation of the plane tangent to the graph of

$$f(x, y) = \frac{x^2 + y^2}{xy},$$

   at $(x_0, y_0) = (1, 2)$.

3. Let $v(r, t) = t^n \mathrm{e}^{-r^2/(4t)}$. Find a value of the constant n such that v satisfies the following equation,

$$\frac{\partial v}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right).$$

   (An equation like the above is referred to as a partial differential equation or PDE.)

4. Find a parametrisation of the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ in terms of sines and cosines. How does the parametrisation change when the ellipse is translated to have its centre at $(9, -4, 0)$?