§2.6, 3.1 NAME: ____________________

Spring MATH 2220

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1 Gradients and Directional Derivatives

1. Given a function $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, its *gradient* is
$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$
2. The directional derivative of f in the direction of a *unit* vector $\mathbf{v}$ at the point $\mathbf{x}$ is
$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x} + t\mathbf{v})|_{t=0} = \nabla f(\mathbf{x}) \cdot \mathbf{v}$$
3. The direction $\mathbf{v}$ in which f is increasing the fastest at $\mathbf{x}$ is the direction parallel to $\nabla f(x)$. The direction of fastest decrease is parallel to $-\nabla f(x)$.
4. For $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ a C^1 function, with $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$, the vector $\nabla f(x_0, y_0, z_0)$ is perpendicular to the level set $f(x, y, z) = f(x_0, y_0, z_0)$. Thus the tangent plane to this level set is
$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

2 Higher-Order Derivatives

1. **Equality of mixed partial derivatives**: If $f(x, y)$ is C^2 (has continuous 2nd partial derivatives) then
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$
This is sometimes referred to as the Clairaut-Schwarz theorem.

3 Practice Problems

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function in two variables. Define the function $F(x, y, z) = z - f(x, y)$ (clearly $F : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 function in three variables). Find a formula for the tangent plane to the level set $F(x, y, z) = 0$. Does this formula seem similar to you?
2. Find a vector $\mathbf{v}(x, y, z)$ normal to the surface
$$z = \sqrt{x^2 + y^2} + (x^2 + y^2)^{3/2}$$

at a general point $(x, y, z) \neq (0,0,0)$.

Find the cosine of the angle between $\mathbf{v}(x, y, z)$ and the z-axis and determine the limit of $\cos\theta$ as $(x, y, z) \to (0,0,0)$.

3. Compute $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ for $u = \cos(xy^2)$ and verify that they are equal.

4. Let $w = f(x, y)$ be a function of two variables and let

$$x = u + v \quad , \quad y = u - v.$$

Show that

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}$$

5. The two equations $\mathrm{e}^u \cos v = x$ and $\mathrm{e}^u \sin v = y$ define u and v as functions of x and y say $u = U(x, y)$ and $v = V(x, y)$. Find explicit formulae for U and V. Show that $\nabla U(x, y)$ and $\nabla V(x, y)$ are perpendicular at each point (x, y).

6. Find a constant c such that at any point of intersection of the two spheres

$$(x - c)^2 + y^2 + z^2 = 3 \quad \text{and} \quad x^2 + (y-1)^2 + z^2 = 1$$

will be perpendicular to each other.