§3.3, 3.5 NAME: ______________________

Spring MATH 2220 19 February 2020

1 Extrema of Real Valued functions

1. A *local minimum point* of $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a point $\mathbf{x}_0 \in U$ such that $f(\mathbf{x}_0) \leq f(\mathbf{x})$ for all $\mathbf{x}$ in some neighbourhood of $\mathbf{x}_0$. The value of f at that point, $f(\mathbf{x}_0)$ is called the *local minimum value*. By replacing '$\leq$' in the above definition by '$\geq$' we get the definition of *local maximum point* and *value*.

2. **The First Derivative Test:** If f is differentiable in an open set U and $\mathbf{x} \in U$ is a local extremum, then $\mathbf{x}_0$ is a *critical point* i.e.

$$\frac{\partial f}{\partial x_1}(\mathbf{x}_0) = ... = \frac{\partial f}{\partial x_n}(\mathbf{x}_0) = 0.$$

2 The Implicit Function Theorem

1. **One variable version:** If $f : (a, b) \to \mathbb{R}$ is C^1 and if $f'(x_0) \neq 0$, then locally near x_0, f has an inverse function $x = f^{-1}(y)$. Moreover f^{-1} is C^1. We think of x as the known/independent variable and y as the unknown/dependent variable.

2. **Special n-variable Version:** If $F : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^1 and at a point $(\mathbf{x}_0, z_0) \in \mathbb{R}^{n+1}$, $F(\mathbf{x}_0, z_0) = 0$ and $\frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0$, then locally near $(\mathbf{x}_0, z_0)$ there is a unique solution $z = g(\mathbf{x})$ of the equation $F(\mathbf{x}, z) = 0$. We say that $F(\mathbf{x}, z) = 0$ *implicitly defines* z as a function of $\mathbf{x} = (x_1, ..., x_n)$. Moreover g is C^1. Here, we think of $x_1, ..., x_n$ as the known/independent variables and z as the unknown/dependent variable.

3. The partial derivatives are computed by implicit differentiation:

$$\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x_i} = 0,$$

and so

$$\frac{\partial z}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z}.$$

4. In other words, if $\nabla g(\mathbf{x}_0) \neq 0$, then the level set $g = c$ is a smooth surface near $\mathbf{x}_0$.

5. The General version of the Implicit function theorem deals with functions $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$. Component-wise we may write the following:

$$F_1(x_1, ..., x_n, z_1, ..., z_m) = 0$$
$$\vdots$$
$$F_m(x_1, ..., x_n, z_1, ..., z_m) = 0.$$

for m unknown variables $\mathbf{z} = (z_1, ..., z_m)$. The theorem states that if the Jacobian of the function in the unknown variable at $(\mathbf{x}_0, z_0)$ is non-zero, viz.,

$$J_z(f)(\mathbf{x}_0, \mathbf{z}_0) = \begin{vmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial z_1} & \cdots & \frac{\partial F_m}{\partial z_m} \end{vmatrix} \neq 0$$

then these equations locally define $(z_1, ..., z_m)$ as functions of $(x_1, ..., x_n)$.

6. **Inverse function theorem:** If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 mapping, then the condition for inverting the function near a given point $\mathbf{x}_0$, $\mathbf{y}_0 = f(\mathbf{x}_0)$ is

$$J_x(f)(\mathbf{x}_0) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{vmatrix} \neq 0.$$

7. When we say a statement is **locally** true near a point it means that we can find an **open set** around that point where the statement is true.

3 Practice Problems

1. Discuss the solvability of the following systems:

   (a)

$$\begin{aligned} 3x + 2y + z^2 + u + v^2 &= 0 \\ 4x + 3y + u^2 + v + w + 2 &= 0 \\ x + z + w + u^2 + 2 &= 0 \end{aligned}$$

   for u, v, w in terms of x, y, z near $x = y = z = 0$, $u = v = 0$ and $w = -2$.

   (b)

$$\begin{aligned} u(x, y, z) &= x + xyz \\ v(x, y, z) &= y + xy \\ w(x, y, z) &= z + 2x + 3z^2 \end{aligned}$$

   for x, y, z in terms of u, v, w near $(x, y, z) = (0, 0, 0)$.

2. Let u be defined as a function of x and y by means of the equation

$$u = F(x + u, yu).$$

   Find $\partial u/\partial x$ and $\partial u/\partial y$ in terms of the partial derivatives of F (in some region where u is implicitly defined).

3. Where does solvability of z fail for this equation:

$$\sin(x + y) + \sin(y + z) = 1.$$

   Suppose we can solve for z as $z = f(x, y)$ in some open set. Compute the derivative $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of x, y, z. Also try to compute $\frac{\partial^2 f}{\partial x \partial y}$.