§3.3 NAME: ______________________

Spring MATH 2220

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1 Extrema of Real Valued functions

1. A *local minimum point* of $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a point $\mathbf{x}_0 \in U$ such that $f(\mathbf{x}_0) \leq f(\mathbf{x})$ for all $\mathbf{x}$ in some neighbourhood of $\mathbf{x}_0$. The value of f at that point, $f(\mathbf{x}_0)$ is called the *local minimum value*. By replacing '$\leq$' in the above definition by '$\geq$' we get the definition of *local maximum point* and *value*.

2. **The First Derivative Test:** If f is differentiable in an open set U and $\mathbf{x} \in U$ is a local extremum, then $\mathbf{x}_0$ is a *critical point* i.e.

$$\frac{\partial f}{\partial x_1}(\mathbf{x}_0) = ... = \frac{\partial f}{\partial x_n}(\mathbf{x}_0) = 0.$$

3. Just as we generalise first derivatives of multi-variable functions as gradients (which are vectors), the appropriate generalisation of the second derivative is a matrix referred to as the *Hessian*:

$$Hf(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

Note that the Hessian is a symmetric matrix if $f \in C^2$.

4. **The Second Derivative Test:** If $\mathbf{x}_0$ is a critical point of the function $f : \mathbb{R}^n \to \mathbb{R}$, and

- $Hf(\mathbf{x}_0)$ is positive definite or equivalently, all its eigenvalues are strictly positive, the $\mathbf{x}_0$ is a local minimum.
- On the other hand if the matrix is negative definite i.e. its eigenvalues are strictly negative, then $\mathbf{x}_0$ is a maximum.
- If there are a mixture of negative and positive eigenvalues then $\mathbf{x}_0$ is a saddle point.
- In every other case, the test is inconclusive.

5. **For functions of Two variables:** A point (x_0, y_0) is a local minimum if,

(i) $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$,

(ii) $\frac{\partial^2 f}{\partial^2 x}(x_0, y_0) > 0$,

(iii) and

$$D := \det \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial^2 y}. \end{bmatrix} > 0.$$

If (i) and (iii) hold but (ii) is modified to be $\frac{\partial^2 f}{\partial^2 x}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum.
If the *discriminant*, D is negative, then (x_0, y_0) is a saddle point.

6. **Global extrema:** Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$, where A need not be open. A point $\mathbf{x}_0 \in A$ is a *global minimum* if $f(\mathbf{x}_0) \leq f(\mathbf{x})$ for all $\mathbf{x} \in A$. The definition for a global maximum is obtained by reversing the inequality.

7. **Extreme value theorem:** Let $A \subset \mathbb{R}^n$ be **closed** (contains its boundary) and **bounded** (you can draw a large enough ball that fully contains A). If $f : A \to \mathbb{R}$ is continuous, then f has at least one global maximum and at least one global minimum. A subset of $\mathbb{R}^n$ which is closed and bounded is called *compact* (A in the above theorem is compact).

8. **Strategy for Global extrema:** To find the global extrema on a closed and bounded i.e. on a set $D = U \cup \partial U$:

   (a) find the critical points in U

   (b) find the maximum/minimum points of f on ∂U

   (c) compute the value of f at the points you found in (i) and (ii)

   (d) the largest of these gives the global maximum and the smallest gives the global minimum.

2 Practice Problems

1. Find the points on the elliptic paraboloid $z = 4x^2 + y^2$ to $(0, 0, 8)$.

2. Find the Global extrema of the function $f(x, y) = 5x^2 - 2y^2 + 10$ on the disk $x^2 + y^2 \leq 1$.

3. A rectangular box, open at the top, is to hold 256 cubic centimetres of sand. Find the dimensions for which the surface area (which includes the bottom and the four sides) is minimized.

4. Write the number 120 as a sum of three numbers so that the sum of the products taken two at a time is a maximum.

5. *Method of least Squares.* Given n distinct numers $x_1, ..., x_n$ and n further numbers $y_1, ..., y_n$ (not necessarily distinct), it is generally impossible to find a straight line $f(x) = ax + b$ which passes through all the points (x_i, y_i) that is such that $f(x_i) = y_i$ for each i. However we can try a linear function which makes the "total squared error"

$$E(a, b) = \sum_{i=1}^{n} [f(x_i) - y_i]^2$$

   a minimum. Determine the values of a and b which do this.