§3.4, 5.1, 5.2 NAME: ____________________

Spring MATH 2220

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1 Constrained Extrema

1. **Lagrange Multipliers:** Let $f \in U \subset \mathbb{R}^n \to \mathbb{R}$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}$ be C^1. We consider the problem of extremising f on level set of of g, say $g(\mathbf{x}) = c$. If $\mathbf{x}_0$ is such an extremum and if $\nabla g(\mathbf{x}_0) \neq 0$ then:

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0).$$

2. If there are k constraints,

$$g_1 = c_1,\ ...,\ g_k = c_k$$

then the Lagrange multiplier Equation is

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g(\mathbf{x}_0) + ... + \lambda_k \nabla g(\mathbf{x}_0)$$

3. For $n = 2$, to test whether an extremum you found using Lagrange multipliers is a local maximum or local minimum calculate the determinant of the *Bordered Hessian Matrix*:

$$\left|\bar{H}\right| = \det \begin{bmatrix} 0 & -\partial g/\partial x & -\partial g/\partial y \\ -\partial g/\partial x & \partial^2 h/\partial x^2 & \partial^2 h/\partial x \partial y \\ -\partial g/\partial y & \partial^2 h/\partial x \partial y & \partial^2 h/\partial y^2 \end{bmatrix}.$$

If $\left|\bar{H}\right| > 0$ $\mathbf{x}_0$ is a local maximum and if $\left|\bar{H}\right| < 0$ it is a maximum. Otherwise the test is inconclusive.

2 The Riemann integral over Rectangular domains

1. A *Riemann sum* for a function f on a domain $R = [a, b] \times [c, d]$ has the form:

$$S_n = \sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk}) \Delta x \Delta y,$$

where R is divided into n^2 equal sub-rectangles or cells and $\mathbf{c}_{jk}$ is a point chosen in the jk^{th} sub-rectangle, $0 \leq j, k \leq n - 1$ of width Δx and height Δy.

2. Definition of the Integral: If $\lim_{n\to\infty} S_n = S$ exists and is independent of the choice of $\mathbf{c}_{jk}$, f is said to be *integrable* over R and the limit is denoted

$$\iint_R f(x, y) \mathrm{d}A \quad \text{or} \quad \iint_R f(x, y) \mathrm{d}x \mathrm{d}y.$$

3. Functions that are continuous or bounded (with discontinuities along the finite union of graphs of functions) on the rectangle are integrable.

4. The integral is linear in its argument and (sub-)additive with respect to its region. It also satisfies the triangle inequality:

$$\left|\iint_R f \mathrm{d}A\right| \leq \iint_R |f| \,\mathrm{d}A.$$

5. *Fubini's theorem* states that if f is continuous the reduction to an iterated integral holds:

$$\iint_R f\mathrm{d}A = \int_a^b \left[\int_c^d f(x,y)\mathrm{d}y\right] \mathrm{d}x = \int_c^d \left[\int_a^b f(x,y)\mathrm{d}x\right] \mathrm{d}y.$$

3 Practice Problems

1. Find the points on the curve of intersection of the two surfaces:

$$x^2 - xy + y^2 - z^2 = 1 \quad \text{and} \quad x^2 + y^2 = 1$$

nearest to the origin.

2. Find the minimum volume bounded by the planes $x = 0$, $y = 0$, $z = 0$ and a plane that is tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at a point in the positive x, y, z octant.

3. Find the maximum of $\log x + \log y + 3\log z$ on that portion of the sphere $x^2+y^2+z^2 = 5r^2$ where $x > 0$, $y > 0$, $z > 0$.

4. Compute the volume of the solid bounded by the surface $z = \sin y$, the planes $x = 1$, $x = 0$, $y = 0$ and $y = \pi/2$ and the xy-plane.

5. Evaluate

$$\int_0^1 \int_0^1 (xye^{x+y})\mathrm{d}y\mathrm{d}x$$

6. Let f be continuous on $R = [a,b] \times [c,d]$. For $a < x < b$ and $c < y < d$, define

$$F(x,y) = \int_a^x \int_c^y f(u,v)\mathrm{d}v\mathrm{d}u.$$

Show that,

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = f(x,y)$$