§5.3, 5.4 Name: ____________________

Spring MATH 2220

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1 Double Integrals Over More General Regions

1. **Elementary Regions:** A *y-simple region* is one that lies between two continuous curves $y = \phi_1(x)$ and $y = \phi_2(x)$ where $\phi_1(x) \leq \phi_2(x)$ and $a \leq x \leq b$. Similarly an *x-simple region* is one that lies between two continuous curves $x = \psi_1(y)$ and $x = \psi_2(y)$ where $\psi_1(y) \leq \psi_2(y)$ and $c \leq y \leq d$. An *elementary region* is one that is either y-simple or is x-simple. If it is both, we say the region is *simple*.

2. The integral of a function f over an elementary region D is obtained by extending f to f^* a function on the a rectangle R that contains D, the function defined to be

$$f^*(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D \text{ and } (x,y) \in R \end{cases}$$

and the integral is defined to by

$$\iint_D f \mathrm{d}A = \iint_R f^* \mathrm{d}A.$$

3. For a y-simple region:

$$\iint_D f \mathrm{d}A = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y) \mathrm{d}y \mathrm{d}x$$

and for an x-simple region:

$$\iint_D f \mathrm{d}A = \int_c^d \int_{\psi_1(x)}^{\psi_2(x)} f(x,y) \mathrm{d}x \mathrm{d}y$$

4. If D is a *simple* region and if f is integrable on D then,

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y) \mathrm{d}y \mathrm{d}x = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y) \mathrm{d}x \mathrm{d}y$$

5. $m \leq f(x,y) \leq M$ on an elementary region D, then the *mean value inequality* holds:

$$m \mathrm{Area}(D) \leq \iint_D f \mathrm{d}A \leq M \mathrm{Area}(D)$$

6. If f is continuous and D is an elementary region then the *mean value equality* holds

$$\iint_D f(x,y) \mathrm{d}A = f(x_0, y_0) \mathrm{Area}(D)$$

for some point $(x_0, y_0) \in D$.

2 Practice Problems

1. Sketch the region and change the order of integration:
   (a) $\int_1^e \int_0^{\log x} f(x,y)\mathrm{d}y\mathrm{d}x$
   (b) $\int_0^\pi \int_{-\sin(x/2)}^{\sin(x)} f(x,y)\mathrm{d}y\mathrm{d}x$

2. Evaluate the integral:
$$\int_0^1 \int_y^1 \sin(x^2)\mathrm{d}x\mathrm{d}y$$

3. Derive the formula:
$$\int_0^a \int_0^y \mathrm{e}^{m(a-x)} f(x)\mathrm{d}x\mathrm{d}y = \int_0^a (a-x)\mathrm{e}^{m(a-x)}\mathrm{d}x$$

4. When a double integral was set up for the volume of the solid under the paraboloid $z = x^2 + y^2$ and above a region S of the xy-plane, the following sum of iterated integrals was obtained:
$$V = \int_0^1 \int_0^y (x^2+y^2)\mathrm{d}x\mathrm{d}y + \int_1^2 \int_0^{2-y} (x^2+y^2)\mathrm{d}x\mathrm{d}y.$$
   Sketch the region of integration on the xy-plane and write the iterated integral with the order reversed. Also compute the integral.

5. If $f(x,y) = \mathrm{e}^{\sin(x+y)}$ and $D = [-\pi,\pi] \times [-\pi,\pi]$, show that
$$\frac{1}{e} \leq \frac{1}{4\pi^2} \iint_D f(x,y)\mathrm{d}A \leq e.$$