§5.5, 1.4, 6.1* Name: ____________________

Spring MATH 2220
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11 March 2020

1 Triple integrals

1. If f is an integrable function defined on a box $B = [a,b] \times [c,d] \times [p,q] \subset \mathbb{R}^3$, the triple integral can be calculated as:

$$\iiint_B f \, \mathrm{d}V = \int_a^b \left[\int_c^d \left[\int_p^q f(x,y,z) \, \mathrm{d}z \right] \mathrm{d}y \right] \mathrm{d}x$$

2. An **Elementary region,** $W \subset \mathbb{R}^3$ is defined by the inequalities $a \leq x \leq b$, $\varphi_1(x) \leq y \leq \varphi_2(x)$ and $\gamma_1(x,y) \leq z \leq \gamma_2(x,y)$. The integral of a function in this region is calculated by

$$\iiint_W f \, \mathrm{d}V = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$$

2 Cylindrical and Spherical Polar coordinates

1. The **polar coordinates** (r, θ) of a point (x,y) in the xy-plane are given by

$$x = r\cos\theta \quad y = r\sin\theta.$$

The inverse map to convert (x,y) to (r,θ):

$$r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

2. The **cylindrical coordinates** (r,θ,z) of a point $(x,y,z) \in \mathbb{R}^3$ are determined by

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z.$$

The inverse map to convert (x,y,z) to (r,θ,z):

$$r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad z = z.$$

Volume element in cylindrical coordinates:

$$\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z.$$

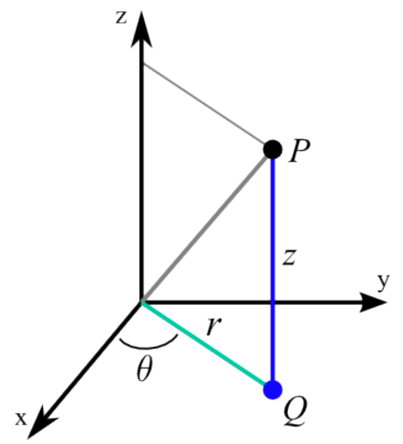

Figure 1: Cylindrical coordinates of the point P. Image credits: mathinsight.org

3. The **Spherical polar coordinates** (ρ, θ, ϕ) of a point $(x, y, z) \in \mathbb{R}^3$ are

$$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \cos\theta \quad z = \rho \cos\phi.$$

The inverse map to convert (x, y, z) to (r, θ, z):

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).$$

Spherical polar coordinates are exactly like specifying the longitude (ϕ) and latitude (θ) of a point along with its distance from the centre of the Earth (ρ).

Volume element in cylindrical coordinates:

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

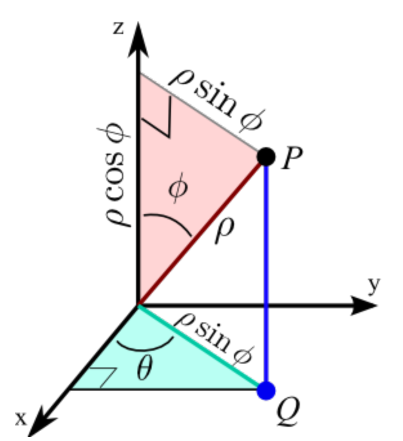

Figure 2: Spherical coordinates of the point P. Image credits: mathinsight.org

3 Practice Problems

1. Evaluate the integral
$$\iiint_{\mathcal{W}} xyz\,dV,$$
for the region $\mathcal{W}$ defined by $0 \le z \le 1$, $0 \le y \le \sqrt{1-x^2}$, $0 \le x \le 1$.

2. Find the volume of a solid in the octant $x \le 0$, $y \le 0$, $z \le 0$ bounded by $x + y + z = 1$ and $x + y + 2z = 1$.

3. Find the equation in spherical polar coordinates of a cylinder along the z-axis lying between $0 \le z \le h$ and radius R.

4. Find the equation in cylindrical coordinates of a sphere of radius R.

5. Determine the mass of a solid lying between two concentric spheres of radii a and b where $0 < a < b$, if the density at each point is equal to the square of the distance of this point from the centre.
Hint: The mass of a solid lying the region $\mathcal{W} \subset \mathbb{R}^3$ of density $f(x, y, z)$ is given by $M = \iiint_{\mathcal{W}} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.