§6.1, 6.2, Review Name: ____________________

Spring MATH 2220
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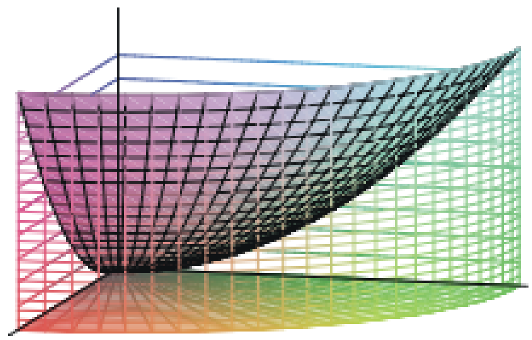
1 Maps from between $\mathbb{R}^n$ and change of variables

1. A map $T : D^* \to \mathbb{R}^2$ associates every point $(u, v) \in D^* \subset \mathbb{R}^2$ to a point $T(u, v) = (x, y) \in \mathbb{R}^2$. The *image* of the map T is denoted $D = T(D^*)$.
2. A *linear* map is one that can be represented as a 2×2 matrix. Remember that linear maps take parallelograms to parallelograms and they map the sides and vertices of the first to those of the second.
3. A map T is $1 - 1$ or *injective* if different points in the domain are sent to different points in the range or in other words, if (u, v) and (u', v') are mapped onto the same point, then $(u, v) = (u', v')$ viz, $T(u, v) = T(u', v') \implies (u, v) = (u', v')$.
4. When D the image of T, we say that T maps D^* onto D or is *surjective*.
5. The Jacobian matrix of a C^1 map,
$$\begin{aligned} T : D^* \subset \mathbb{R}^2 &\to \mathbb{R}^2 \\ (u, v) &\mapsto (x(u, v), y(u, v)) \end{aligned}$$
is
$$J = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$
and we denote the determinant of this matrix
$$\det(J) = \frac{\partial(x, y)}{\partial(u, v)}.$$
6. The two variable change of variables formula states that for a C^1 map $\tau : D^* \to D$ that is 1-1 and onto D and an integrable function $f : D \to \mathbb{R}$,
$$\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \mathrm{d}u\,\mathrm{d}v.$$
Note that there is an absolute sign on the Jacobian determinant.
7. The above statements for $\mathbb{R}^2$ can be generalised easily to mappings between $\mathbb{R}^n$.

2 Practice Problems

1. Find the volume of the the region between the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ contained in the $x \geq 0$, $y \geq 0$ and $z \geq 0$ octant by choosing an appropriate coordinate system.



   Now express the same volume but in the following order of integration $\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y$.

2. Express the volume bounded between the ellipsoid $(x/4)^2 + (y/6)^2 + (z/4)^2 = 1$ and the plane $z/4 + y/6 = 1$ in the following orders of integration:

   (a) $\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$

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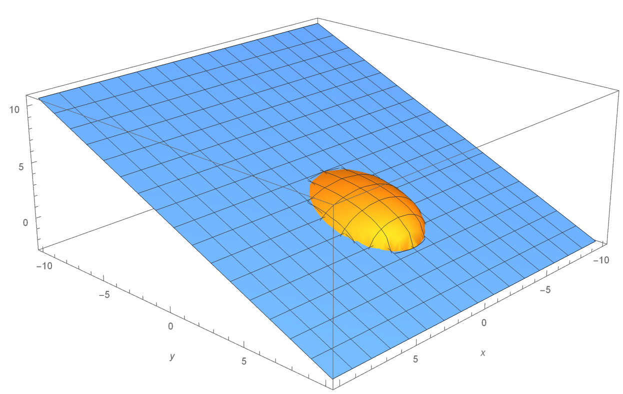

3. Consider the mapping T defined by the equations

$$x = u + v \qquad y = v - u^2$$

   (a) Calculate the Jacobian determinant $\frac{\partial(x,y)}{\partial(u,v)}$.

(b) A triangle denoted $\triangle$ in th uv-plane has corners at $(0,0)$, $(0,2)$ and $(2,0)$. Sketch the image $S := T(\triangle)$ in the xy plane.
Hint: Try to see how the mapping acts on the sides of the triangle.

(c) Express the integral to compute the area of S in the xy plane.

(d) Evaluate $\iint_S (x-y+1)^{-2}\,\mathrm{d}x\,\mathrm{d}y$ using change of coordinates.

4. Evaluate $\iint_R \frac{y}{x}\,\mathrm{d}x\,\mathrm{d}y$ where R is the region having boundaries $x^2-y^2=1$, $x^2-y^2=4$ $y=0$ and $y=x/2$ using an appropriate change of coordinates.

5. Consider the mapping T defined by the equations $x = u^2 - v^2$ and $y = 2uv$.

(a) Compute the Jacobian.

(b) Let R be the rectangle in the uv-plane with corners at (1,1), (2,1), (2,3) and (1,3). Sketch the image of R under the map T.

(c) What region in the uv-plane maps onto the region $B = \{(x,y) : x^2+y^2 \leq 1\}$, the unit disk in the xy-plane?

(d) Evaluate $\iint_C xy\,\mathrm{d}x\,\mathrm{d}y$ by using the coordinate change.