§4.2, 4.3, 7.1, 7.2 NAME: ______________________

Spring MATH 2220

TA: Gokul Nair

22 April 2020

1 Integrals over curves

1. The *path integral* of a scalar valued function $f : \mathbb{R}^3 \to \mathbb{R}$ along a C^1 path parametrised as$\mathbf{c}(t)$, $a \leq t \leq b$ is
$$\int_{\mathbf{c}} f \, \mathrm{d}s = \int_a^b f(\mathbf{c}(t)) \, \|\mathbf{c}'(t)\| \, \mathrm{d}t.$$
2. Setting $f = 1$ in the above integral gives the *arc length* of the path.
3. The *line integral* of a continuous vector field $\mathbf{F}\mathbb{R}^3 \to \mathbb{R}^3$ along a C^1 path parametrised as$\mathbf{c}(t)$, $a \leq t \leq b$ is
$$\int_{\mathbf{c}} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \, \mathrm{d}t.$$
4. A version of the fundamental theorem relating line integrals to gradients: If $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 scalar field then
$$\int_{\mathbf{c}} \nabla f \cdot \, \mathrm{d}\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

2 Practice Problems

1. Compute the line integral of $\mathbf{F}(x, y) = \left(\frac{1}{|x|+|y|}, \frac{1}{|x|+|y|}\right)$, along the square with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ in an anti-clockwise direction.
2. Compute the line integral
$$\int_{\mathbf{c}} y \, \mathrm{d}x + z \, \mathrm{d}y + x \, \mathrm{d}z,$$
where $\mathbf{c}$ is the intersection of the two surfaces $z = xy$ and $x^2 + y^2 = 1$ in an anti-clockwise direction when viewed from the $+z$ axis.
3. Evaluate the path integral of $f(x, y, z) = \mathrm{e}^{\sqrt{z}}$ along the path $\mathbf{c}(t) = (1, 2, t^2)$, where $0 \leq t \leq 1$.
4. Suppose $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field that has the property that $\int_{\mathbf{c}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = 0$ for every **closed** path $\mathbf{c}$. Then prove that if $\mathbf{c}_1$ and $\mathbf{c}_2$ are two different paths with the **same** starting and ending points, then
$$\int_{\mathbf{c}_1} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s}.$$