§7.3,7.4 Name: ____________________________

Spring MATH 2220

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1 Surfaces and Surface integrals

1. Curves can be parametrised using one parameter. Surfaces on the other hand need two parameters to be described.

2. A *parametrised surface* is a map

$$\begin{aligned}\mathbf{\Phi} :& D \to \mathbb{R}^3\\ &(u,v) \mapsto (x(u,v), y(u,v), z(u,v))\end{aligned}$$

3. The surface itself is the image of the map, $\mathbf{\Phi}(D)$.

4. Tangent vectors to the surface are given by

$$\mathbf{\Phi}_u = \frac{\partial \mathbf{\Phi}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

and

$$\mathbf{\Phi}_v = \frac{\partial \mathbf{\Phi}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right),$$

with a normal vector

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v.$$

5. The area of a surface is given by

$$A(S) = \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\| \, \mathrm{d}u \, \mathrm{d}v$$

6. The scalar surface area element is given by

$$\mathrm{d}S = \|\mathbf{T}_u \times \mathbf{T}_v\| \, \mathrm{d}u \, \mathrm{d}v,$$

and

7. For a sphere $x^2 + y^2 + z^2 = R^2$, the element is $\mathrm{d}S = R^2 \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta$.

8. The natural parameterisation for the graph of a function $z = g(x,y)$ is

$$\mathbf{\Phi}(x,y) = (x, y, g(x,y)),$$

where we use x and y themselves as the parameters. The scalar surface area element is

$$\mathrm{d}S = \left[\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\right] \mathrm{d}x \, \mathrm{d}y.$$

2 Practice Problems

1. Consider the surface in $\mathbb{R}^3$ parametrised by

$$\mathbf{\Phi}(r,\theta) = (r\cos\theta, r\sin\theta, \theta) \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

   (a) Sketch and describe the surface.

   (b) Show that the surface is regular and find an expression for the unit normal to the surface.

   (c) Find the equation for the plane tangent to the surface at a point (x_0, y_0, z_0) on it.

   (d) Compute the mass of the surface given that the mass density at a point (x, y, z) is equal to twice the distance of (x, y, z) to the central axis.

2. Compute the area of the region cut from the plane $x + y + z = a$ by the cylinder $x^2 + y^2 = a^2$.

3. Compute the area of that portion of the conical surface $x^2 + y^2 = z^2$ which lies above the xy-plane and is cut off by the sphere $x^2 + y^2 + z^2 = 2ax$.