§7.6, 8.1, 8.2, 8.3, 8.4 Name: ____________________

Spring MATH 2220

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1 Fundamental theorems of vector calculus

1. Let $\mathbf{F}$ be a vector field defined on S, the image of a parametrised surface $\mathbf{\Phi}$. The surface integral of $\mathbf{F}$ is

$$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v) \, \mathrm{d}u \, \mathrm{d}v.$$

2. **Green's theorem** Consider a simple region D with bounding curve ∂D. If P and Q are C^1 functions on with domain D, we have

$$\int_{\partial D} P \, \mathrm{d}x + Q \, \mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y.$$

3. **Stokes' theorem** Let S be an oriented surface defined by a one-to-one parametrisation $\mathbf{\Phi} : D \subset \mathbb{R}^2 \to S$, where D is a region to which Green's theorem applies. Let ∂S denote the oriented boundary of S and let $\mathbf{F}$ be a C^1 vector field on S. Then

$$\iint_S (\nabla \times F) \cdot \mathrm{d}\mathbf{S} = \int_{\partial S} F \cdot \mathrm{d}s.$$

4. If S has no boundary i.e. $\partial S = \emptyset$, then $\int_{\partial S} F \cdot \mathrm{d}s = 0$, so the Stokes' theorem reads

$$\iint_S (\nabla \times F) \cdot \mathrm{d}\mathbf{S} = 0.$$

5. For a C^1 vector field, F defined on all of $\mathbb{R}^3$ all of these statements are equivalent:

   (a) The line integral of F around any closed loop is zero.

   (b) $\mathbf{F} = \nabla f$ for some scalar field $f(x, y, z)$.

   (c) $\nabla \times \mathbf{F} = 0$.

   Such a vector field is said to be conservative.

6. **Divergence theorem** Let $W \subset \mathbb{R}^3$ be a region with boundary ∂W oriented by the outward pointing unit normal. $\mathbf{F}$ is a smooth vector field defined on W, then

$$\iiint_W (\nabla \cdot F) \, \mathrm{d}V = \iint_{\partial W} \mathbf{F} \cdot \mathrm{d}S.$$

2 Practice Problems

1. Consider $D = \{(x,y) | x^2 + y^2 < 0\}$, the open unit disk. Let

$$P(x,y) = \frac{y}{x^2+y^2} \qquad Q(x,y) = \frac{-x}{x^2+y^2}.$$

   (a) Compute the line integral

$$\int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y$$

   (b) Apply Green's theorem and compute the relevant integral.

   (c) Do you notice an issue?

2. Use Green's theorem to evaluate the line integral $\int_C y^2\,\mathrm{d}x + x\,\mathrm{d}y$ when

   (a) C is a square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.

   (b) C is the circle of radius 2 centred at the origin.

   (c) C is parametrised as $\mathbf{r}(t) = 2\cos^3 t\mathbf{i} + 2\sin^3 t\mathbf{j}$ $0 \le t \le 2\pi$.

3. Let C be the curve $x^2+y^2 = 1$ $z = 0$ and let S be the surface S_1 together with S_2 where S_1 is defined by $x^2 + y^2 \le 1$, $z = -1$ and S_2 is defined by $x^2 + y^2 = 1$, $-1 \le z \le 0$.

   (a) Draw a figure showing an orientation such that Stokes' theorem applies the the surface S and the curve C.

   (b) If R is another surface with boundary C, show that

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}S = \iint_R (\nabla \times \mathbf{F}) \cdot \mathrm{d}S$$

   (c) If $F(x,y,z) = (y^3 + \mathrm{e}^{xz})\mathbf{i} - (x^3 + \mathrm{e}^{yz})\mathbf{j} + \mathrm{e}^{xyz}\mathbf{k}$, calculate

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}S,$$

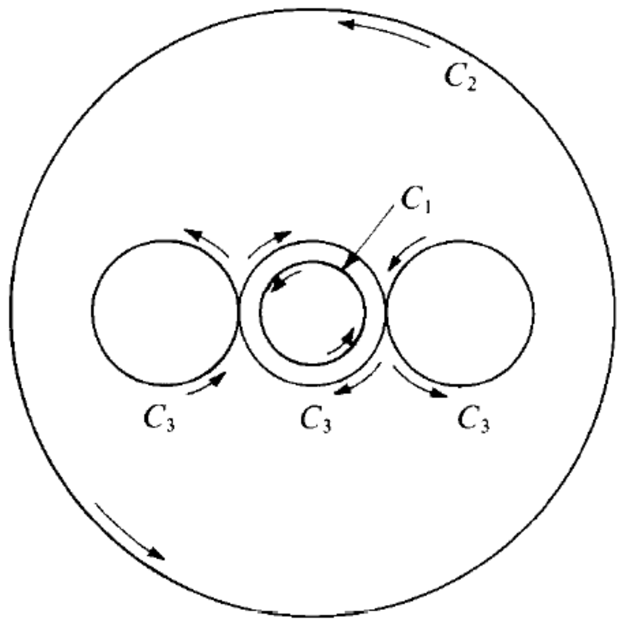
   where S is the surface given above.

4. Let $I_k = \int_{C_k} P\,\mathrm{d}x + Q\,\mathrm{d}y$, where

$$P(x,y) = -y\left[\frac{1}{(x-1)^2+y^2} + \frac{1}{x^2+y^2} + \frac{1}{(x+1)^2+y^2}\right]$$

$$Q(x,y) = \frac{x-1}{(x-1)^2+y^2} + \frac{x}{x^2+y^2} + \frac{x+1}{(x+1)^2+y^2}$$

   and C_k are the circles shown in the following figure:



C_1 is the smallest circle $x^2+y^2 = 1/8$ traced anticlockwise, C_2 is the largest, $x^2+y^2 = 4$ traced anticlockwise and C_3 is the curve made up of the three intermediate circles $(x-1)^2 + y^2 = 1/4$, $x^2 + y^2 = 1/4$ and $(x+1)^2 + y^2 = 1/4$ traced out as shown. If $I_2 = 6\pi$ and $I_3 = 2\pi$, find the value of I_1.