§1.1,1.2,1.3 **Name:** ____________

Fall MATH 2930

02 September 2021

Review

Differential Equations

An ordinary differential equation can be written generally as

$$F\left(t, y, \frac{\mathrm{d}y}{\mathrm{d}t}, ..., \frac{\mathrm{d}^n y}{\mathrm{d}t^n}\right) = 0$$

Classification of Differential equations

- Number of independent variables: **Ordinary** or **Partial**,

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}t} + y = \sin(t) \quad \text{vs} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial x \partial y} + u^3 = 0$$

- Number of dependent variables: **Single** variable or **System** of variables,

$$t\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0 \quad \text{vs} \quad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x - xy \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = xy - y \end{cases}$$

- **Order**: Highest derivative occurring in the DE,

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}t} + y = \sin(t) \quad \text{vs} \quad \frac{\mathrm{d}^3y}{\mathrm{d}t^3} + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + y = 0$$

- **Linear** or **Nonlinear**: Depending on whether F is a linear or nonlinear function y and its derivatives,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0 \quad \text{vs} \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \sin(y) = 0$$

Practice Problems

1. Is the function $y(t) = 1 + t$ a solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{y^2 - 1}{t^2 + 2t}$? Why or why not? What about $y = 1 + 2t$? What about $y = 1$?

   Solution: To check whether these functions are solutions, we plug the given expressions for $y(t)$ into both sides of the differential equation and check whether they are equal.

   For $y(t) = 1 + t$, we get $\frac{dy}{dt} = \frac{d}{dt}\big[1 + t\big] = 1$ and

   $$\frac{y^2 - 1}{t^2 + 2t} = \frac{(1+t)^2 - 1}{t^2 + 2t} = \frac{t^2 + 2t}{t^2 + 2t} = 1$$

   since these two quantities are equal, $y(t) = 1+t$ is a solution to the differential equation.

   For $y(t) = 1 + 2t$, we get $\frac{dy}{dt} = 2$, but $\frac{y^2-1}{t^2+2t} = \frac{4t^2+4t}{t^2+2t} \neq 2$, so this is not a solution.

   For $y(t) = 1$, we get $\frac{dy}{dt} = 0$ and $\frac{y^2-1}{t^2+2t} = 0$, so this is another solution.

2. Suppose you and a friend are interested in how the number of fish in Lake Cayuga changes over time. Luckily, you're both in this class, and you decide to use a differential equation as a model. You remember that one of the tenets of math modelling is to start with a simple model and then build up complexity as needed, so you agree to make the following simplifying assumptions:

   (a) there is only one species of fish in the lake;

   (b) the species reproduces continuously;

   (c) the species has access to unlimited resources (eg: food, space);

   (d) the species is not being preyed upon;

   (e) no fish are being added to the lake artificially (eg: fishy immigration).

   You friend has come up with a few candidate models for $P(t)$, denoting the population size at time t. For each equation, discuss whether or not it is a good model in this context, specifically in light of the above assumptions as well as normal physical constraints (eg: there can't be a negative number of fish).

   (a) $\frac{dP}{dt} = (t+1)^2$
   Solution: This equation would imply that the rate of growth of the population depends directly on the time, when, in fact, it should depend indirectly on time via the number of fish currently in the lake, $P(t)$, because fish reproduce from other fish not based on the time on the clock. Thus, this model is unsuitable.

   (b) $\frac{dP}{dt} = 2P$
   Solution: This equation implies that, for every fish parent, the population increases by two new fish babies per unit time, in a continuous fashion, which agrees

with assumption (ii). Also, the right-hand side of this equation is positive for positive P (i.e. for physically relevant population sizes), so P is always increasing and we are not violating assumptions (iii) or (iv). Lastly, when $P = 0$, $dP/dt = 0$, so if the fish go extinct they remain extinct, satisfying assumption (v). Thus, this model seems to be a good candidate, based on our assumptions. We will see later in the course that this type of population model is unrealistic because it implies perpetual exponential growth of the population.

(c) $\frac{dP}{dt} = 1 - P^2$

Solution: This equation implies that when $P > 1$, the population size is decreasing, but assumptions (iii) and (iv) specifically state that there are unlimited resources and that the fish are not being preyed on. Since we're assuming that there is no reason for the population to decline, this model is unsuitable.

(d) $\frac{dP}{dt} = P^2 + 1$

Solution: While the rate of change of the population is always positive (so the population is always increasing), it may seem reasonable at first glance based on our assumptions. However, if we check $P = 0$, $dP/dt = 1$, which would imply that when there are no fish, the population is still growing at the rate of 1 population unit per unit time, which violates assumption (v). Thus, this model is not suitable either.

3. Modelling Predator-Prey relationships:

   (a) Draw a graph with the population of fish, $P(t)$ on the y axis and time on the x axis. **Solution:** $P(t)$ increases (exponentially) with time.

   (b) Suppose at $t = t_0$, that is after allowing some time to elapse, we introduce a species of shark into the lake. Let the number of sharks at time t be $Q(t)$. The sharks begin to prey on the fish already in the lake. Draw a graph showing how $P(t)$ would behave for a short period of time after t_0. **Solution:** We expect $P(t)$ to decrease as the fish are hunted by the sharks.

   (c) As the sharks continue to prey on the fish, they reproduce in larger numbers. Draw a graph showing how $Q(t)$ would behave for a short period of time after t_0. **Solution:** As the sharks reproduce, we expect $Q(t)$ to increase and to look similar to that $P(t)$ before the sharks were introduced.

   (d) Can the shark population grow without limits? What happens when there are too many sharks and too few fish to prey on? Draw a graph representing both $P(t)$ and $Q(t)$ over a large period of time. **Solution:** When there are too many sharks and too few fish, we expect the shark population to decrease as they die of starvation. As the shark population dwindles, the fish will have an easier time reproducing and increasing in population. But this will not happen indefinitely– the fish and shark populations will oscillate as time progresses.

   (e) How would a graph with P on the x axis and Q on the y axis look? **Solution:** A closed loop.

4. Consider the differential equation

$$\frac{dy}{dt} = \frac{1}{2}\, y(2+y)(y-8).$$

   For what values of y is a solution to this differential equation

   (a) increasing,

   (b) decreasing,

   (c) constant?

   In each case, explain your reasoning.

   Solution: Whether a (differentiable) function is increasing, decreasing, or constant is equivalent to its derivative being positive, negative, or zero, respectively. Thus, we can simply analyze the sign of the right-hand side of the equation for various values of y to answer the question, which can be done easily by plotting the right-hand side of the equation.

Some of the questions in this worksheet were prepared by Elliot Cartee.