§2.1,2.2,2.3 **Name:** ____________________

Fall MATH 2930 09 September 2021

Review

Integrating factors

- A linear first order differential equation can generally be written as

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = g(t). \tag{1}$$

- The integrating factor for (1) is given by

$$\mu(t) = \exp\left(\int p(t)dt\right).$$

- The general solution to (1) is

$$y(t) = \frac{1}{\mu(t)}\left(\int \mu(s)g(s)ds + c\right).$$

Separable ODEs

- An ODE is separable if it can be written as

$$N(y)\frac{\mathrm{d}y}{\mathrm{d}x} + M(x) = 0.$$

- The solution is implicitly given by

$$\int N(y)dy + \int M(x)dx = 0.$$

- Sometimes it is possible to solve the resulting equation to obtain y as a function of x.

Practice Problems

1. Free fall with air resistance
   Recall that Newton's 2nd law says that $mass \times acceleration = Force$, i.e.

   $$m\frac{\mathrm{d}v}{\mathrm{d}t} = F.$$

   (a) Suppose a ball of mass m is falling under the influence of gravity (a constant force with magnitude mg). Assume that there is no air resistance and write down the ODE for the velocity v of the particle. **Solution:** $dv/dt = g$

   (b) Now suppose that there is air resistance. The force due to air resistance is proportional to the velocity of the ball and in the opposite direction of motion. Write down the ODE for the velocity v of the particle. **Solution:** $mdv/dt = mg - \eta v$

   (c) We conduct an experiment by dropping a $2kg$ ball in air with no initial velocity. Assume that the friction constant (constant of proportionality) is 1 kg/s. What is the velocity of the ball after 5 seconds. Take $g = 10m/s^2$. **Solution:** The integrating factor is given by $\mu(t) = e^{\int 1/2dt}$. Solving the equation yields $20 - 20e^{-t/2}$.

2. Logistic growth
   When unlimited resources are available to a species, their population tends to grow exponentially with time. This behaviour can be described by the following differential equation;

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = rP,$$

   where r is referred to as the reproduction rate.

   We now try to understand what happens when there is a limit to the amount of resources available.

   (a) One way of modelling the limitation of resources is by saying that the population growth rate, $\frac{\mathrm{d}P}{\mathrm{d}t}$ is positive as long as P is less than some maximum size, K and if P crosses K, $\frac{\mathrm{d}P}{\mathrm{d}t}$ becomes negative. Choose the ODE with this property:

   i. $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{rP}{K-P}$

   ii. $\frac{\mathrm{d}P}{\mathrm{d}t} = 1 - r\left(\frac{P}{K}\right)^2$

   iii. $\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{K}\right)$

   Hint: More than one of the options may have the required property, but only one really makes sense

(b) Solve the one that you chose with $r = 1$ and $K = 1$. Assume that at $t = 0$, $P = 1/2$ (ask the TA for a hint if you are finding it hard to integrate). **Solution:** The equation is separable.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(1-P) \implies \frac{dP}{P(1-P)} = dt$$

Use $\frac{1}{P(1-P)} = \frac{1}{P} + \frac{1}{1-P}$ to get $\ln P - \ln(1-P) = t$. Solving for P yields $P(t) = Ce^t/(1+Ce^t)$. Plugging in the initial conditions, we get $C = 1$.

(c) Seasonal variations : Now suppose that the reproduction rate of the species depends on the season. Since the season is periodic we can naively model it by a *cos* function. One way of incorporating this into our original model is to multiply the right-hand-side of the ODE by $(1 + q\cos(t))$ where q is a constant. Solve the resulting ODE again assuming $r = 1$, $K = 1$, and $P(0) = 1/2$. **Solution:** The equation is still separable, and the solution is $P(t) = \frac{Ce^{t+q\sin t}}{1+Ce^{t+q\sin t}}$. Again the initial conditions give $C = 1$.

(d) Plot the solutions of (b) and (c) on a computer. For (c) use $q = 1$, $q = 10$ and $q = 100$ and see how the behaviour changes. (If no one in your group has a computer with them, you can do this part at home)

3. Solve the following ODEs:

   (a) $t^3y' + 4t^2y = e^{-t}$ **Solution:** The integrating factor is given by $\mu(t) = \exp(4\ln t) = t^4$. Solving the ODE, we get $y(t) = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{C}{t^4}$.

   (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x - e^{-x}}{y + e^y}$ **Solution:** This is separable, and the solution is given implicitly by $y^2 - x^2 + 2(e^y - e^x) = C$.

4. Consider the initial value problem

$$y' - 3y = 3t + e^{2t} \quad y(0) = y_0$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$? **Solution:** The integrating factor is e^{-3t}, and solving the ODE we get $y = -t - \frac{1}{3} - e^{-2t} + \left(y_0 + \frac{4}{3}\right)e^{3t}$. As $t \to \infty$ the exponential dominates the long-term behaviour. Thus the critical value is $y_0 = \frac{-4}{3}$. At exactly the critical value, the linear term dominates, so $y \to -\infty$.