§2.3,2.4,2.5 **Name:** ____________________

Fall MATH 2930 16 September 2021

Review

Autonomous ODEs

- An *autonomous* ODE can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y).$$

- An *equilibrium* or *critical point* of the equation is a y_* such that $f(y_*) = 0$.
- Let y_* be an equilibrium then:
  - y_* is *asymptotically stable* if solutions $y(t)$ starting close enough to y_* converge to y_* as $t \to \infty$.
  - y_* is *unstable* if solutions $y(t)$ do not converge to y_* as $t \to \infty$ even if they start very close to y_*.
- The stability of y_* can be determined using a *phase line*. Take y to be close to y_* and check:
  - Stable: If $f(y) < 0$ for $y > y_*$ and $f(y) > 0$ for $y < y_*$
  - Unstable: If $f(y) > 0$ for $y > y_*$ and $f(y) < 0$ for $y < y_*$
  - Semistable: Stable on one side and unstable on the other.

Practice Problems[1]

1. Consider the initial value problem

$$y' - 3y = 3t + e^{2t} \quad y(0) = y_0$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$? **Solution:** The integrating factor is e^{-3t}, and solving the ODE we get $y = -t - \frac{1}{3} - e^{-2t} + \left(y_0 + \frac{4}{3}\right) e^{3t}$. As $t \to \infty$ the exponential dominates the long-term behaviour. Thus the critical value is $y_0 = \frac{-4}{3}$. At exactly the critical value, the linear term dominates, so $y \to -\infty$.

2. For each part below, create a continuous autonomous differential equation that has the stated properties (if possible). If it is not possible, provide justification for why not. *Hint: Try to draw the phase-line first.*

   (a) Exactly three equilibria, two unstable and one stable. **Solution:** $dy/dt = y^3 - y$

   (b) Exactly two equilibria, both stable **Solution:** Impossible

   (c) Exactly two equilibria, one unstable and one semi-stable. **Solution:** $dy/dt = (y-1)y^2$

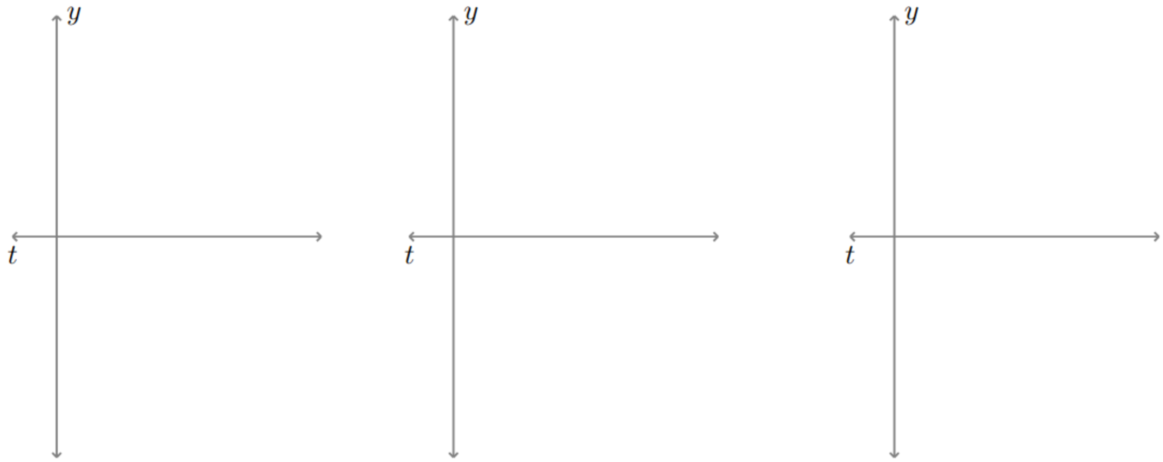
3. Consider the autonomous ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - y^3.$$

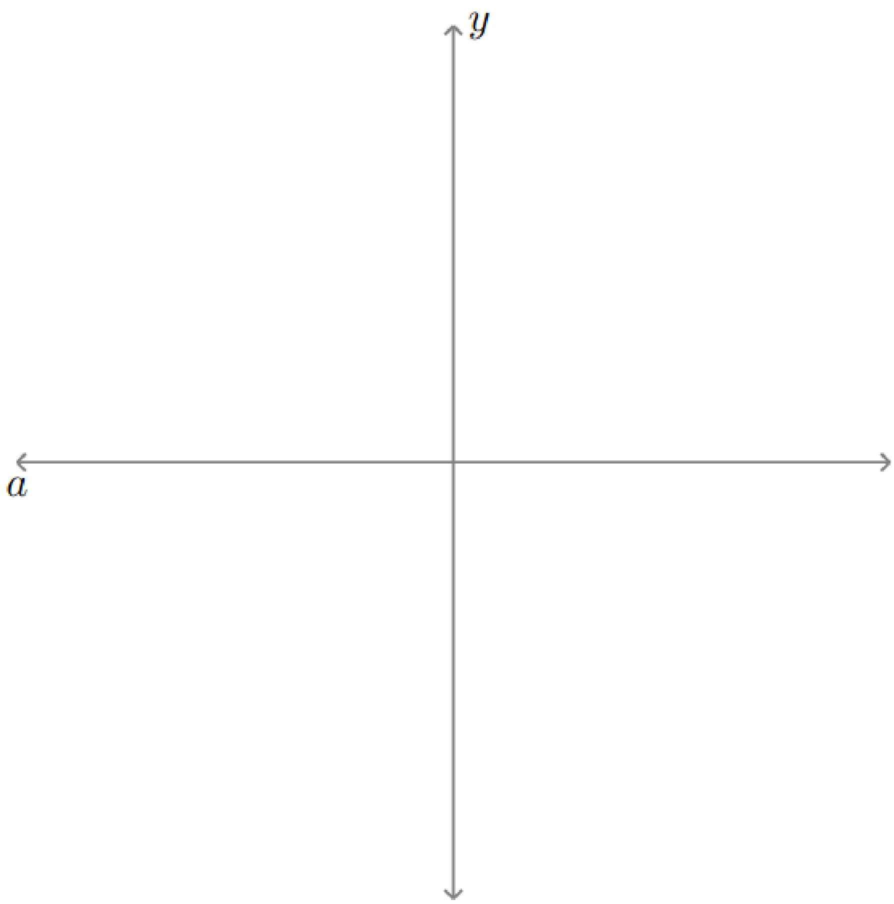
a is a *parameter* and it doesn't depend on y or t. It can either be positive or negative. You will see that the number of equilibria will depend on the value of a.

   (a) On the y-t axes below, sketch the qualitative behaviour of solutions $y(t)$ for a couple different values of a.

[1]Problems in this worksheet are adapted from `https://math.uchicago.edu/~ecartee` and `https://iode.wordpress.ncsu.edu/` which is distributed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International Licence



(b) (Optional) On the a-y axes below, plot the different equilibria y versus the value of the parameter a. Label which parts of the graph correspond to stable and unstable equilibria.



4. Solve the following ODEs:

   (a) $t^3y' + 4t^2y = e^{-t}$ **Solution:** The integrating factor is given by $\mu(t) = \exp(4\ln t) = t^4$. Solving the ODE, we get $y(t) = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{C}{t^4}$.

   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - e^{-x}}{y + e^y}$ **Solution:** This is separable, and the solution is given implicitly by $y^2 - x^2 + 2(e^y - e^x) = C$.

5. (a) Let C be the temperature (in Fahrenheit) of a cup of coffee that is cooling off to room temperature (70 Fahrenheit). Which of the following differential equations best models the situation and why? *Hint: What do you think should be the equilibrium (or equilibria) temperature(s)? What about stability?*

   i. $C' = 0.4(C - 70)$

   ii. $C' = 0.4(1 - C/70)$

   iii. $C' = -0.4C + 28$

   iv. $C' = -0.4C$

   v. $C' = -0.4(C - 70)$

   vi. $C' = -0.4(C - 28)$

   vii. $C' = -0.4(C - 70)^2$

   Solution: Want $C = 70$ to be the unique equilibrium and stable. Only iii and v work.

   (b) For the equation you picked in part (a), draw a phase line. Now sketch some solutions for different initial values on the C-t plane. Be sure to label any equilibria and their stability. **Solution:**

   (c) Consider a situation where there are two separate cups of coffee. At $t = 0$, one of the cups starts at a temperature of $C = 180$, while the other starts at a temperature of $C = 160$.

   Will there ever be a time t at which the temperatures of the two cups are *exactly* the same? Why or why not? **Solution:** In this case, the two cups of coffee will both asymptotically approach the equilibrium temperature of 70 degrees, but there is no point in time at which they will have exactly the same temperature. One way of thinking about this is that being the same temperature would mean that these solutions cross in the C-t plane, which should never happen. Another way is that these two solutions are in fact horizontal translations of each other. This can be understood by thinking that the cup of coffee starting at $C(0) = 160$ is somehow ahead of the other by some fixed amount of time. In order for the cup starting at $C(0) = 180$ to reach some fixed temperature, say 140, then it would take whatever time it takes to reach 160 + however long it took the cup of coffee starting at 160 to reach 140, so it will always be slower. This horizontal translation idea can also be seen by separating variables. Solutions will satisfy $\int \frac{dC}{-0.4(C-70)} = \int dt = t + K$, where the solutions are distinguished by the value of K, which can be seen here as a horizontal translation.