§2.3,2.4,2.5 NAME: ____________________

Fall MATH 2930 16 September 2021

REVIEW

AUTONOMOUS ODES

- An *autonomous* ODE can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y).$$

- An *equilibrium* or *critical point* of the equation is a y_* such that $f(y_*) = 0$.

- Let y_* be an equilibrium then:
  - y_* is *asymptotically stable* if solutions $y(t)$ starting close enough to y_* converge to y_* as $t \to \infty$.
  - y_* is *unstable* if solutions $y(t)$ do not converge to y_* as $t \to \infty$ even if they start very close to y_*.

- The stability of y_* can be determined using a *phase line*. Take y to be close to y_* and check:
  - Stable: If $f(y) < 0$ for $y > y_*$ and $f(y) > 0$ for $y < y_*$
  - Unstable: If $f(y) > 0$ for $y > y_*$ and $f(y) < 0$ for $y < y_*$
  - Semistable: Stable on one side and unstable on the other.

Practice Problems[1]

1. Consider the initial value problem

$$y' - 3y = 3t + e^{2t} \quad y(0) = y_0$$

   Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

2. For each part below, create a continuous autonomous differential equation that has the stated properties (if possible). If it is not possible, provide justification for why not. *Hint: Try to draw the phase-line first.*

   (a) Exactly three equilibria, two unstable and one stable.

   (b) Exactly two equilibria, both stable

   (c) Exactly two equilibria, one unstable and one semi-stable.

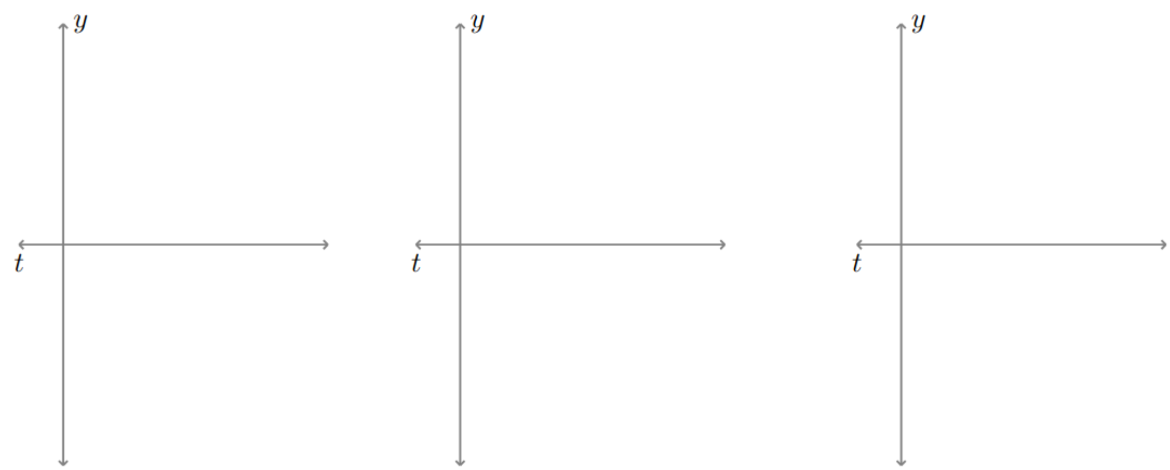
[1]Problems in this worksheet are adapted from `https://math.uchicago.edu/~ecartee` and `https://iode.wordpress.ncsu.edu/`

3. Consider the autonomous ODE:

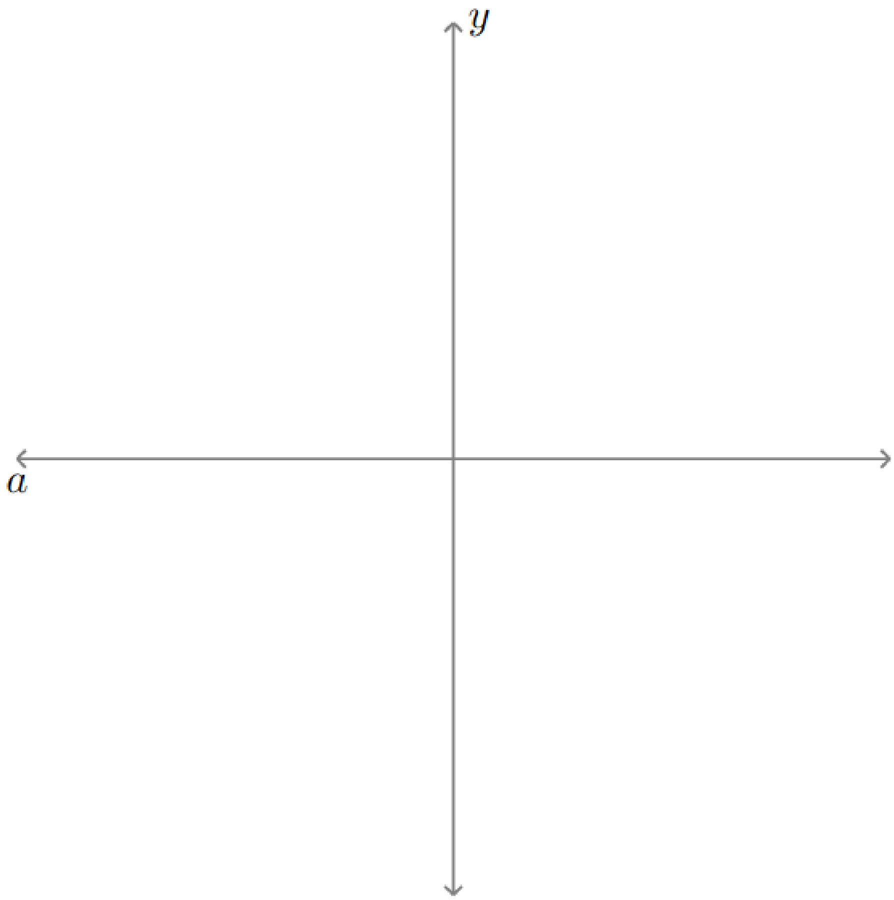
$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - y^3.$$

a is a *parameter* and it doesn't depend on y or t. It can either be positive or negative. You will see that the number of equilibria will depend on the value of a.

(a) On the y-t axes below, sketch the qualitative behaviour of solutions $y(t)$ for a couple different values of a.



(b) (Optional) On the a-y axes below, plot the different equilibria y versus the value of the parameter a. Label which parts of the graph correspond to stable and unstable equilibria.



4. Solve the following ODEs:

   (a) $t^3y' + 4t^2y = e^{-t}$

   (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x - e^{-x}}{y + e^y}$

5. (a) Let C be the temperature (in Fahrenheit) of a cup of coffee that is cooling off

to room temperature (70 Fahrenheit). Which of the following differential equations best models the situation and why? *Hint: What do you think should be the equilibrium (or equilibria) temperature(s)? What about stability?*

i. $C' = 0.4(C - 70)$

ii. $C' = 0.4(1 - C/70)$

iii. $C' = -0.4C + 28$

iv. $C' = -0.4C$

v. $C' = -0.4(C - 70)$

vi. $C' = -0.4(C - 28)$

vii. $C' = -0.4(C - 70)^2$

(b) For the equation you picked in part (a), draw a phase line. Now sketch some solutions for different initial values on the C-t plane. Be sure to label any equilibria and their stability.

(c) Consider a situation where there are two separate cups of coffee. At $t = 0$, one of the cups starts at a temperature of $C = 180$, while the other starts at a temperature of $C = 160$.

Will there ever be a time t at which the temperatures of the two cups are *exactly* the same? Why or why not?