§2.6,2.7 NAME: ______________________

Fall MATH 2930 23 September 2021

REVIEW

EXACT ODES

- The differential equation $M(x,y) + N(x,y)y' = 0$ is *exact* if there is a function $\psi(x,y)$ such that:

$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y) \quad \frac{\partial \psi}{\partial y}(x,y) = N(x,y).$$

- Solutions to the above exact equation are given *implicitly* by

$$\psi(x,y) = c.$$

- If M and N are continuously differentiable, then the equation being exact is equivalent to having

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

- If the equation is not separable, but $\frac{M_y - N_x}{N}$ is dependent *only* on x (doesn't depend on y), then you can find an integrating factor, μ by solving

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{M_y - N_x}{N}\mu$$

NUMERICAL INTEGRATION

- For the differential equation $y' = f(y,t)$, Euler integration can be written iteratively as

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n).$$

- Usually we take the difference $(t_{n+1} - t_n)$ to be some fixed number, h, which is also referred to as the *step size*.

Practice Problems[1]

1. Consider the differential equation

$$(xy^2 + bx^2y) + (x + y)x^2y' = 0$$

   (a) Find the value(s) of b for which the given equation is exact.
   Solution: We solve for $M_y = N_x$, thus

$$\frac{\partial}{\partial y}[xy^2 + bx^2y] = \frac{\partial}{\partial x}[(x + y)x^2].$$

   Simplifying, we get $2xy + bx^2 = 3x^2 + 2xy$, so $b = 3$.

   (b) Solve the equation for the value(s) of b you found.
   Solution: Let $\psi(x, y)$ be the the function such that $\psi_x = M = xy^2 + 3x^2y$ and $\psi_y = N = x^3 + x^2y$. Integrating the first equation with respect to x while holding y constant, we get

$$\psi(x, y) = \frac{x^2y^2}{2} + x^3y + f(y),$$

   for some function $f(y)$. Differentiating this with respect to y and setting it equal to N, we have

$$x^2y + x^3 + f'(y) = x^3 + x^2y$$

   so clearly $f'(y) = 0$ and $f(y)$ is just a constant. Thus the solution is implicitly given by

$$\frac{x^2y^2}{2} + x^3y = C.$$

2. Consider the equation

$$\frac{dy}{dt} = \lambda y$$
$$y(0) = y_0$$

   (a) Write down the solution to this equation (by now, you've solved this equation several times).
   Solution: $y(t) = y_0e^{\lambda t}$.

   (b) Now, instead of solving it directly, write down three Euler iterates using a step size h.
   Solution:

$$y_1 = y_0 + h\lambda y_0 = (1 + h\lambda)y_0,$$
$$y_2 = y_1 + h\lambda y_1 = y_1(1 + h\lambda) = (1 + h\lambda)^2y_0,$$
$$y_3 = y_2 + h\lambda y_2 = y_2(1 + h\lambda) = (1 + h\lambda)^3y_0.$$

[1]Some of the Problems are taken from `https://math.uchicago.edu/~ecartee/2930_sp19/worksheet4.pdf`

(c) Find a formula for the n^{th} Euler iterate, y_n. The formula should only depend on y_0, λ and n.

Solution: $y_n = (1 + h\lambda)^n y_0$.

(d) Suppose $y_0 = 1$ and $\lambda = -1$. Are there any values of the step size h that would be totally unacceptable to pick i.e. are there values of h for which the numerical solution behaves very differently from the analytical solution? If so, why?

Solution: For those values of λ and y_0, we have $y_n = (1 - h)^n$. If $h \geq 1$, we see that y_n could become 0 or even negative for some values of n. This would be completely inconsistent with the analytic solution which always remains positive. The reason we are getting this behaviour is because we chose a step size that is too large and our approximation breaks down.

(e) Let $y_0 = 1$ and $\lambda = 1$ and suppose we want to compute the solution via Euler's method in an interval $[0, t]$. Let $n = t/h$ and show that y_n converges to the *analytical* solution as $h \to 0$.

Solution: Letting $n = t/h$,

$$y_n = (1 + h)^{t/h}$$

Then we are interested in taking the limit as $h \to 0$

$$\lim_{h\to 0} y_n = \lim_{h\to 0} (1 + h)^{t/h}$$

To figure out this limit, it helps to take the logarithm of both sides,

$$lim_{h\to 0} \ln(y_n) = \lim_{h\to 0} \ln\left[(1 + h)^{t/h}\right] = \lim_{h\to 0} \frac{t\ln(1 + h)}{h}$$

this can then be evaluated using L'hôpital's rule,

$$\lim_{h\to 0} \ln(y_n) = \lim_{h\to 0} \frac{t\frac{1}{1+h}}{1} = t$$

Then since

$$\lim_{h\to 0} \ln(y_n) = t$$

we have

$$\lim_{h\to 0} y_n = e^t$$

matching our solution from part

3. Consider the differential equation

$$x^2y^3 + x(1 + y^2)y' = 0.$$

(a) Show that the equation is <u>not</u> exact.

Solution: For this equation:

$$M(x, y) = x^2y^3, \qquad N(x, y) = x(1 + y^2)$$

To check if it's exact, we calculate

$$\frac{\partial M}{\partial y} = 3x^2y^2$$

and

$$\frac{\partial N}{\partial x} = 1 + y^2$$

$$M_y \neq N_x, \quad \text{so this equation is not exact.}$$

(b) Show that it can be made exact by multiplying both sides of the equation by the integrating factor $\mu(x, y) = \frac{1}{xy^3}$.

Solution: Multiplying everything by the integrating factor $\mu(x, y) = \frac{1}{xy^3}$,

$$M(x, y) = x$$

$$\frac{\partial M}{\partial y} = 0$$

and

$$N(x, y) = \frac{1}{y^3} + \frac{1}{y}$$

$$\frac{\partial N}{\partial x} = 0$$

Since $M_y = N_x$, the equation

$$x + \left(\frac{1}{y^3} + \frac{1}{y}\right) y' = 0, \quad \text{is exact.}$$

(in fact it is also separable).

(c) Now that the equation is exact, solve it.

Solution: We want to find a function $\psi(x, y)$ with partial derivatives:

$$\frac{\partial \psi}{\partial x} = x, \qquad \frac{\partial \psi}{\partial y} = \frac{1}{y^3} + \frac{1}{y}$$

Integrating the first of those, we get

$$\psi(x, y) = \frac{1}{2}x^2 + f(y)$$

for some function $f(y)$. Integrating the second equation,

$$\psi(x, y) = \frac{-1}{2y^2} + \ln(y) + g(x)$$

Combining these two, we find that the solution is:

$$\psi(x, y) = \frac{1}{2}x^2 + \frac{-1}{2y^2} + \ln(y) = C$$

4. Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t + 1 \quad y(0) = 1$$

(a) Approximate $y(0.1)$, $y(0.2)$ and $y(0.3)$ using Euler's method with $h = 0.1$.

Solution:

$$y_1 = y_0 + (0 + 1) \times h = 1 + 0.1 = 1.1$$

$$y_2 = y_1 + (0.1 + 1) \times h = 1.1 + (1.1 \times 0.1) = 1.22$$

$$y_3 = y_2 + (0.2 + 1) \times h = 1.22 + (1.3 \times 0.1) = 1.35$$

(b) Solve the equation analytically and compute $y(0.1)$, $y(0.2)$ and $y(0.3)$ using the actual solution. Are you satisfied with the approximation?