§2.6,2.7 **Name:** ______________________

Fall MATH 2930 23 September 2021

Review

Exact ODEs

- The differential equation $M(x,y) + N(x,y)y' = 0$ is *exact* if there is a function $\psi(x,y)$ such that:
$$\frac{\partial \psi}{\partial x}(x,y) = M(x,y) \quad \frac{\partial \psi}{\partial y}(x,y) = N(x,y).$$
- Solutions to the above exact equation are given *implicitly* by
$$\psi(x,y) = c.$$
- If M and N are continuously differentiable, then the equation being exact is equivalent to having
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$
- If the equation is not separable, but $\frac{M_y - N_x}{N}$ is dependent *only* on x (doesn't depend on y), then you can find an integrating factor, μ by solving
$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{M_y - N_x}{N}\mu$$

Numerical Integration

- For the differential equation $y' = f(y,t)$, Euler integration can be written iteratively as
$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n).$$
- Usually we take the difference $(t_{n+1} - t_n)$ to be some fixed number, h, which is also referred to as the *step size*.

Practice Problems[1]

1. Consider the differential equation

$$(xy^2 + bx^2y) + (x+y)x^2y' = 0$$

   (a) Find the value(s) of b for which the given equation is exact.

   (b) Solve the equation for the value(s) of b you found.

2. Consider the equation

$$\frac{dy}{dt} = \lambda y$$
$$y(0) = y_0$$

   (a) Write down the solution to this equation (by now, you've solved this equation several times).

   (b) Now, instead of solving it directly, write down three Euler iterates using a step size h.

[1]Some of the Problems are taken from `https://math.uchicago.edu/~ecartee/2930_sp19/worksheet4.pdf`

(c) Find a formula for the n^{th} Euler iterate, y_n. The formula should only depend on y_0, λ and n.

(d) Suppose $y_0 = 1$ and $\lambda = -1$. Are there any values of the step size h that would be totally unacceptable to pick i.e. are there values of h for which the numerical solution behaves very differently from the analytical solution? If so, why?

(e) Let $y_0 = 1$ and $\lambda = 1$ and suppose we want to compute the solution via Euler's method in an interval $[0, t]$. Let $n = t/h$ and show that y_n converges to the *analytical* solution as $h \to 0$.

3. Consider the differential equation

$$x^2y^3 + x(1 + y^2)y' = 0.$$

(a) Show that the equation is <u>not</u> exact.

(b) Show that it can be made exact by multiplying both sides of the equation by the integrating factor $\mu(x, y) = \frac{1}{xy^3}$.

(c) Now that the equation is exact, solve it.

4. Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t + 1 \quad y(0) = 1$$

(a) Approximate $y(0.1)$, $y(0.2)$ and $y(0.3)$ using Euler's method with $h = 0.1$.

(b) Solve the equation analytically and compute $y(0.1)$, $y(0.2)$ and $y(0.3)$ using the actual solution. Are you satisfied with the approximation?