§3.1,3.2,3.3 **Name:** ____________________

Fall MATH 2930 30 September 2021

Review

2nd Order ODEs

- A *Linear, Second order, Constant Coefficient, Homogeneous* ODE is written as

$$ay'' + by' + cy = 0 \quad y(t_0) = y_0 \quad y'(t_0) = y_0'$$

- Steps to solve the above:

  1. Plug in $y = e^{rt}$ to get the *Characteristic equation*:

  $$ar^2 + br + c = 0.$$

  2. Solve the characteristic equation (either factoring or using the quadratic formula) to get two roots: r_1 and r_2 (these can potentially be complex numbers)
  3. If $r_1 \neq r_2$, the *general solution* is given by:

  $$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

  4. If $r_1 = r_2$ (repeated roots), the *general solution* is given by

  $$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_2 t}.$$

  5. Plug in the <u>two</u> initial conditions to find C_1 and C_2.

Complex Numbers

- A complex number is of the form $z = a + ib$ where $i = \sqrt{-1}$.
- *Euler's formula* is given by

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

- The exponential of a complex number maybe written as

$$e^z = e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b)$$

Practice Problems

1. Consider the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Find the solution of this initial value problem.

Solution: Plugging in e^{rt} gives the characteristic equation $r^2 + r - 2 = 0$. The roots of this quadratic are -2 and 1, so the general solution is $y(t) = C_1 e^t + C_2 e^{-2t}$. Plugging in the first initial condition gives $1 = C_1 + C_2$. To plug in the second initial condition, we need to compute y'. This is given by $y'(t) = C_1 e^t - 2C_2 e^{-2t}$. Plugging in the second initial condition, we have $1 = C_1 - 2C_2$. Solving the two linear equations for C_1 and C_2 gives $C_1 = 1$ and $C_2 = 0$. Thus the solution is $y(t) = e^t$.

2. Can you find a differential equation whose general solution is

$$y = c_1 e^t + c_2 e^{-4t} \quad ?$$

Solution: We can try to reverse engineer the characteristic equation that would give us this general solution. Looking at y, it is immediate that the two roots of the characteristic equation should be $r_1 = 1$ and $r_2 = -4$. Thus, the characteristic equation would be $(r-1)(r+4) = r^2 + 3r - 4$. Comparing with the formula for the characteristic equation for a general 2nd order ODE, we have $a = 1$, $b = 3$ and $c = -4$, thus the ODE is

$$y'' + 3y' - 4y = 0.$$

3. Consider the Euler formula $e^{is} = \cos s + i \sin s$.

   (a) Replacing s with βt, and then multiplying by $e^{\alpha t}$, we get

$$e^{(\alpha + i\beta)t} = e^{\alpha t}\big(\cos(\beta t) + i \sin(\beta t)\big)$$

   Can you find a similar formula for $e^{(\alpha - i\beta)t}$?

   Solution: We have $e^{(\alpha - i\beta)t} = e^{\alpha t} e^{-i\beta t}$. Using the Euler formula for $e^{i(-\beta t)}$, we get

$$e^{\alpha t} e^{-i\beta t} = e^{\alpha t}(\cos(-\beta t) + i \sin(-\beta t)).$$

   Recalling that $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, we have

$$e^{(\alpha - i\beta)t} = e^{\alpha t}(\cos \beta t - i \sin \beta t)$$

(b) Suppose you have two functions

$$A(t) = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big)$$
$$B(t) = e^{\alpha t}\big(\cos(\beta t) - i\sin(\beta t)\big)$$

Simplify the two following expressions:

$$x_1(t) = \frac{A(t) + B(t)}{2}$$

$$x_2(t) = i\frac{A(t) - B(t)}{2}$$

Solution:

$$\begin{aligned} x_1(t) &= \frac{A(t) + B(t)}{2} \\ &= e^{\alpha t}\frac{\cos\beta t + i\sin\beta t + \cos\beta t - i\sin\beta t}{2} \\ &= e^{\alpha t}\cos\beta t \end{aligned}$$

and

$$\begin{aligned} x_2(t) &= i\frac{A(t) - B(t)}{2} \\ &= ie^{\alpha t}\frac{\cos\beta t + i\sin\beta t - \cos\beta t + i\sin\beta t}{2} \\ &= ie^{\alpha t}(i\sin\beta t) = -e^{\alpha t}\sin\beta t \end{aligned}$$

since $i^2 = -1$.

(c) What do you notice about $x_1(t)$ and $x_2(t)$ compared to $A(t)$ and $B(t)$?
Solution: We notice that x_1 and x_2 are real valued but A and B were complex. This is really cool because we can use these formulae to turn complex valued solutions in real valued ones (which are easier to think about and plot etc.)

(d) If $A(t)$ and $B(t)$ were solutions to a differential equation of the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

would $x_1(t)$ and $x_2(t)$ be solutions too? How about $c_1x_1(t) + c_2x_2(t)$ for arbitrary constants c_1 and c_2? **Solution:** Since the equation above is linear, any linear combination of solutions is still a solution. Since we know that A and B are solutions and since x_1 and x_2 are linear combinations of A and B, they are also solutions. This also means that $c_1x_1 + c_2x_2$ are also solutions.

4. Given an initial value problem

$$ay'' + by' + cy = 0, \qquad y(0) = y_0, \qquad y'(0) = v_0$$

Suppose that for some r_1, r_2, the general solution is:

$$y(t) = C_1e^{r_1t} + C_2e^{r_2t}$$

(a) In order to solve the initial value problem, C_1 and C_2 need to solve a system of two linear equations. What is that system of equations?

Solution: First let's differentiate the solution:

$$y'(t) = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t}$$

Now let's plug in the two initial conditions. This gives two equations

$$C_1 + C_2 = y_0$$

$$C_1 r_1 + C_2 r_2 = v_0$$

In order to find C_1 and C_2 we have to simultaneously solve the two linear equations above. This can be written as

$$\begin{pmatrix} 1 & 1 \\ r_1 & r_2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$$

(b) Given $r_1 \neq r_2$, are you always guaranteed to be able to find C_1 and C_2 to solve the initial value problem? If so, will C_1 and C_2 be unique?

Solution: In order for a solution to exist for the above system of linear equations, the matrix has to have non-zero determinant. That is, $1 \cdot r_2 - 1 \cdot r_1 = r_2 - r_1 \neq 0$. Since it is given that $r_1 \neq r_2$, the above equation is soluble for C_1 and C_2 and the solution will be unique.