§3.1,3.2,3.3

Name: ____________________________

Fall MATH 2930

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Review

2^ND Order ODEs

- A *Linear, Second order, Constant Coefficient, Homogeneous* ODE is written as

$$ay'' + by' + cy = 0 \quad y(t_0) = y_0 \quad y'(t_0) = y_0'$$

- Steps to solve the above:

1. Plug in $y = e^{rt}$ to get the *Characteristic equation*:

$$ar^2 + br + c = 0.$$

2. Solve the characteristic equation (either factoring or using the quadratic formula) to get two roots: r_1 and r_2 (these can potentially be complex numbers)
3. If $r_1 \neq r_2$, the *general solution* is given by:

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

4. If $r_1 = r_2$ (repeated roots), the *general solution* is given by

$$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_2 t}.$$

5. Plug in the <u>two</u> initial conditions to find C_1 and C_2.

Complex Numbers

- A complex number is of the form $z = a + ib$ where $i = \sqrt{-1}$.
- *Euler's formula* is given by

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

- The exponential of a complex number maybe written as

$$e^z = e^{a+ib} = e^a e^{ib} = e^a(\cos b + i\sin b)$$

Practice Problems

1. Consider the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

   Find the solution of this initial value problem.

2. Can you find a differential equation whose general solution is

$$y = c_1 e^t + c_2 e^{-4t} \quad ?$$

3. Consider the Euler formula $e^{is} = \cos s + i \sin s$.

(a) Replacing s with βt, and then multiplying by $e^{\alpha t}$, we get

$$e^{(\alpha+i\beta)t} = e^{\alpha t}\big(\cos(\beta t) + i \sin(\beta t)\big)$$

Can you find a similar formula for $e^{(\alpha-i\beta)t}$?

(b) Suppose you have two functions

$$A(t) = e^{\alpha t}\big(\cos(\beta t) + i \sin(\beta t)\big)$$
$$B(t) = e^{\alpha t}\big(\cos(\beta t) - i \sin(\beta t)\big)$$

Simplify the two following expressions:

$$x_1(t) = \frac{A(t) + B(t)}{2}$$

$$x_2(t) = i\frac{A(t) - B(t)}{2}$$

(c) What do you notice about $x_1(t)$ and $x_2(t)$ compared to $A(t)$ and $B(t)$?

(d) If $A(t)$ and $B(t)$ were solutions to a differential equation of the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

would $x_1(t)$ and $x_2(t)$ be solutions too? How about $c_1x_1(t) + c_2x_2(t)$ for arbitrary constants c_1 and c_2?

4. Given an initial value problem

$$ay'' + by' + cy = 0, \qquad y(0) = y_0, \qquad y'(0) = v_0$$

Suppose that for some r_1, r_2, the general solution is:

$$y(t) = C_1e^{r_1t} + C_2e^{r_2t}$$

(a) In order to solve the initial value problem, C_1 and C_2 need to solve a system of two linear equations. What is that system of equations?

(b) Given $r_1 \neq r_2$, are you always guaranteed to be able to find C_1 and C_2 to solve the initial value problem? If so, will C_1 and C_2 be unique?

Problems in this worksheet are adapted from `https://math.uchicago.edu/~ecartee`