§3.1,3.2,3.3,3.4 NAME: ____________________

Fall MATH 2930 07 October 2021

REVIEW

2ND ORDER ODES

- A *Linear, Second order, Constant Coefficient, Homogeneous* ODE is written as
$$ay'' + by' + cy = 0 \quad y(t_0) = y_0 \quad y'(t_0) = y_0'$$
- Steps to solve the above:
  1. Plug in $y = e^{rt}$ to get the *Characteristic equation*:
  $$ar^2 + br + c = 0.$$
  2. Solve the characteristic equation (either factoring or using the quadratic formula) to get two roots: r_1 and r_2 (these can potentially be complex numbers)
  3. If $r_1 \neq r_2$, the *general solution* is given by:
  $$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$
  4. If $r_1 = r_2$ (repeated roots), the *general solution* is given by
  $$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_2 t}.$$
  5. Plug in the two initial conditions to find C_1 and C_2.

COMPLEX NUMBERS

- A complex number is of the form $z = a + ib$ where $i = \sqrt{-1}$.
- *Euler's formula* is given by
$$e^{i\theta} = \cos\theta + i\sin\theta.$$
- The exponential of a complex number maybe written as
$$e^z = e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b)$$

REDUCTION OF ORDER

Given a differential equation,
$$y'' + p(t)y' + q(t)y = 0$$
and a solution $y_1(t)$, then a second solution, $y_2(t) = v(t)y_1(t)$ can be found by solving
$$y_1 v'' + (2y_1' + py_1)v' = 0$$
for v.

Practice Problems

1. Example Problem: Solve the differential equation:

$$5y'' + 2y' + y = 0 \quad y(0) = 2 \quad y'(0) = 1$$

Solution: Solved in class.

2. Suppose $y_1(t) = t^{-1}$ is a solution to the equation

$$2t^2y'' + 3ty' - y = 0 \quad t > 0.$$

Let $y_2(t) = v(t)y_1(t) = \frac{v(t)}{t}$.

(a) Compute y_2' and y_2'' (your answer will involve v, v' and v'').
Solution: $y' = v't^{-1} - vt^{-2}$ and $y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$

(b) Plug y_2, y_2' and y_2'' into the differential equation and simplify to get an ODE for $v(t)$.
Solution: $2tv'' - v' = 0$.

(c) Solve the resulting ODE for $v(t)$ and hence find $y_2(t)$.
Solution: Let $w = v'$ and solve for w to get $w = ct^{1/2}$. So $v(t) = \frac{2}{3}ct^{3/2} + k$, where c and k are arbitrary constants.

3. On the way to Collegetown, a MATH 2930 student found a piece of paper on the ground. Much of the ink was smudged by water but clearly visible was the word "solution" followed by the expression,

$$y(t) = e^{-2t}(C_1 \sin(t) + C_2 \cos(t)).$$

Also visible (very conveniently) was the phrase "2nd order constant coefficient" but the differential equation itself was smudged except for

$$... + 5y = 0$$

Find the differential equation.

Solution: We can deduce that the roots of the characteristic equation are $r = -2 \pm i$. Suppose the characteristic equation were $ar^2 + br + 5 = 0$, then we have

$$-2 \pm i = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 20a}}{2a}$$

Setting the real parts equal to each other and the imaginary parts equal to each other, we get

$$b = 4a \quad i = \frac{\sqrt{b^2 - 20a}}{2a}$$

Squaring the second equation and substituting the first expression, we get the quadratic $-4a^2 = 16a^2 - 20a$ and the roots are $a = 0, 1$. a can't be 0 since then it wouldn't be second order. So $a = 1$ and $b = 4$ i.e.

$$y'' + 4y' + 5y = 0.$$

4. Solve the initial value problem

$$9y'' - 12y' + 4y = 0 \quad y(0) = 2 \quad y'(0) = -1.$$

Solution: The characteristic equation has the repeated root $\frac{2}{3}$. Hence the general solution of the ODE is

$$y(t) = C_1 e^{2t/3} + C_2 t e^{2t/3}.$$

The derivative is $y'(t) = \frac{2C_1}{3} e^{2t/3} + \frac{2C_2}{3} t e^{2t/3} + C_2 e^{2t/3}$. Plugging in the initial conditions,

$$2 = C_1 \quad -1 = \frac{2C_1}{3} + C_2.$$

So we can solve for C_2 to get $C_2 = \frac{-7}{3}$. The solution is therefore

$$y(t) = 2e^{2t/3} - \frac{7}{3} t e^{2t/3}.$$

Problems in this worksheet are adapted from `https://math.uchicago.edu/~ecartee`